Georgia Milestones

Assessment System

Study/Resource Guide

Study/Resource Guide for Students and Parents

Algebra I

The Study/Resource Guides are intended to serve as a resource for parents and students. They contain practice questions and learning activities for the course. The standards identified in the Study/Resource Guides address a sampling of the state-mandated content standards.

For the purposes of day-to-day classroom instruction, teachers should consult the wide array of resources that can be found at www.georgiastandards.org.

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THE GEORGIA MILESTONES ASSESSMENT SYSTEM

Dear Student,

The **Georgia Milestones Algebra I EOC Study/Resource Guide for Students and Parents** is intended as a resource for parents and students.

This guide contains information about the core content ideas and skills that are covered in the course. There are practice sample questions for every unit. The questions are fully explained and describe why each answer is either correct or incorrect. The explanations also help illustrate how each question connects to the Georgia state standards.

In addition, the guide includes activities that you can try to help you better understand the concepts taught in the course. The standards and additional instructional resources can be found on the Georgia Department of Education website, www.georgiastandards.org.

Get ready—open this guide—and get started!

GEORGIA MILESTONES END-OF-COURSE (EOC) ASSESSMENTS

The EOC assessments serve as the final exam in certain courses. The courses are:

English Language Arts

- Ninth Grade Literature and Composition
- American Literature and Composition

Mathematics

- Algebra I
- Geometry
- Analytic Geometry
- Coordinate Algebra

Science

- Physical Science
- Biology

Social Studies

- United States History
- Economics/Business/Free Enterprise

All End-of-Course assessments accomplish the following:

- Ensure that students are learning
- Count as part of the course grade
- Provide data to teachers, schools, and school districts
- Identify instructional needs and help plan how to meet those needs
- Provide data for use in Georgia’s accountability measures and reports

HOW TO USE THIS GUIDE

Let's get started!

First, preview the entire guide. Learn what is discussed and where to find helpful information. Even though the focus of this guide is Algebra I, you need to keep in mind your overall good reading habits.

- Start reading with a pencil or a highlighter in your hand and sticky notes nearby.
- Mark the important ideas, the things you might want to come back to, or the explanations you have questions about. On that last point, your teacher is your best resource.
- You will find some key ideas and important tips to help you prepare for the test.
- You can learn about the different types of items on the test.
- When you come to the sample items, don't just read them, *do* them. Think about strategies you can use for finding the right answer. Then read the analysis of the item to check your work. The reasoning behind the correct answer is explained for you. It will help you see any faulty reasoning in the ones you may have missed.
- For constructed-response questions, you will be directed to a rubric or scoring guide so you can see what is expected. The rubrics provide guidance on how students earn score points, including criteria for how to earn partial credit for these questions. Always do your best on these questions. Even if you do not know all of the information, you can get partial credit for your responses.
- Use the activities in this guide to get hands-on understanding of the concepts presented in each unit.
- With the Depth of Knowledge (DOK) information, you can gauge just how complex the item is. You will see that some items ask you to recall information and others ask you to infer or go beyond simple recall. The assessment will require all levels of thinking.
- Plan your studying and schedule your time.
- Proper preparation will help you do your best!

OVERVIEW OF THE ALGEBRA I EOC ASSESSMENT

ITEM TYPES

The Algebra I EOC assessment consists of **selected-response**, **constructed-response**, and **extended constructed-response** items.

A **selected-response** item, sometimes called a multiple-choice item, is a question, problem, or statement that is followed by four answer choices. These questions are worth one point.

A **constructed-response** item asks a question and you provide a response that you construct on your own. These questions are worth two points. Partial credit may be awarded if part of the response is correct.

An **extended constructed-response** item is a specific type of constructed-response item that requires a longer, more detailed response. These items are worth four points. Partial credit may be awarded.

Strategies for Answering Constructed-Response Items

- Read the question or prompt carefully.
- Think about what the question is asking you to do.
- Add details, examples, or reasons that help support and explain your response.
- Reread your response and be sure you have answered all parts of the question.
- Be sure that the evidence you have provided supports your answer.
- Your response will be scored based on the accuracy of your response and how well you have supported your answer with details and other evidence.

DEPTH OF KNOWLEDGE DESCRIPTORS

Items found on the Georgia Milestones assessments, including the Algebra I EOC assessment, are developed with a particular emphasis on the kinds of thinking required to answer questions. In current educational terms, this is referred to as Depth of Knowledge (DOK). DOK is measured on a scale of 1 to 4 and refers to the level of cognitive demand (different kinds of thinking) required to complete a task, or in this case, an assessment item. The following table shows the expectations of the four DOK levels in detail.

The DOK table lists the skills addressed in each level as well as common question cues. These question cues not only demonstrate how well you understand each skill but also relate to the expectations that are part of the state standards.

<table>
<tr><th colspan="2">Level 1—Recall of Information

Level 1 generally requires that you identify, list, or define. This level usually asks you to recall facts, terms, concepts, and trends and may ask you to identify specific information contained in documents, maps, charts, tables, graphs, or illustrations. Items that require you to “describe” and/or “explain” could be classified as Level 1 or Level 2. A Level 1 item requires that you just recall, recite, or reproduce information.</th></tr>
<tr><th>Skills Demonstrated</th><th>Question Cues</th></tr>
<tr><td>• Make observations
• Recall information
• Recognize formulas, properties, patterns, processes
• Know vocabulary, definitions
• Know basic concepts
• Perform one-step processes
• Translate from one representation to another
• Identify relationships</td><td>• Find
• List
• Define
• Identify; label; name
• Choose; select
• Compute; estimate
• Express
• Read from data displays
• Order</td></tr>
</table>

<table>
<tr><th colspan="2">Level 2—Basic Reasoning

Level 2 includes the engagement (use) of some mental processing beyond recalling or reproducing a response. A Level 2 “describe” and/or “explain” item would require that you go beyond a description or explanation of recalled information to describe and/or explain a result or “how” or “why.”</th></tr>
<tr><th>Skills Demonstrated</th><th>Question Cues</th></tr>
<tr><td>• Apply learned information to abstract and real-life situations
• Use methods, concepts, and theories in abstract and real-life situations
• Perform multi-step processes
• Solve problems using required skills or knowledge (requires more than habitual response)
• Make a decision about how to proceed
• Identify and organize components of a whole
• Extend patterns
• Identify/describe cause and effect
• Recognize unstated assumptions; make inferences
• Interpret facts
• Compare or contrast simple concepts/ideas</td><td>• Apply
• Calculate; solve
• Complete
• Describe
• Explain how; demonstrate
• Construct data displays
• Construct; draw
• Analyze
• Extend
• Connect
• Classify
• Arrange
• Compare; contrast</td></tr>
</table>

Level 3—Complex Reasoning	
Level 3 requires reasoning, using evidence, and thinking on a higher and more abstract level than Level 1 and Level 2. You will go beyond explaining or describing "how and why" to justifying the "how and why" through application and evidence. Level 3 items often involve making connections across time and place to explain a concept or a "big idea."	
Skills Demonstrated	**Question Cues**
• Solve an open-ended problem with more than one correct answer • Create a pattern • Relate knowledge from several sources • Draw conclusions • Make predictions • Translate knowledge into new contexts • Assess value of methods, concepts, theories, processes, and formulas • Make choices based on a reasoned argument • Verify the value of evidence, information, numbers, and data	• Plan; prepare • Predict • Create; design • Generalize • Justify; explain why; support; convince • Assess • Rank; grade • Test; judge • Recommend • Select • Conclude

Level 4—Extended Reasoning	
Level 4 requires the complex reasoning of Level 3 with the addition of planning, investigating, applying significant conceptual understanding, and/or developing that will most likely require an extended period of time. You may be required to connect and relate ideas and concepts *within* the content area or *among* content areas in order to be at this highest level. The Level 4 items would be a show of evidence, through a task, a product, or an extended response, that the cognitive demands have been met.	
Skills Demonstrated	**Question Cues**
• Analyze and synthesize information from multiple sources • Apply mathematical models to illuminate a problem or situation • Design a mathematical model to inform and solve a practical or abstract situation • Combine and synthesize ideas into new concepts	• Design • Connect • Synthesize • Apply concepts • Critique • Analyze • Create • Prove

DEPTH OF KNOWLEDGE EXAMPLE ITEMS

Example items that represent the applicable DOK levels across various Algebra I content domains are provided on the following pages.

All example and sample items contained in this guide are the property of the Georgia Department of Education.

Example Item 1

DOK Level 1: This is a DOK Level 1 item because it asks students to recall information and determine which relationship does not have the properties that fit the definition of a function.

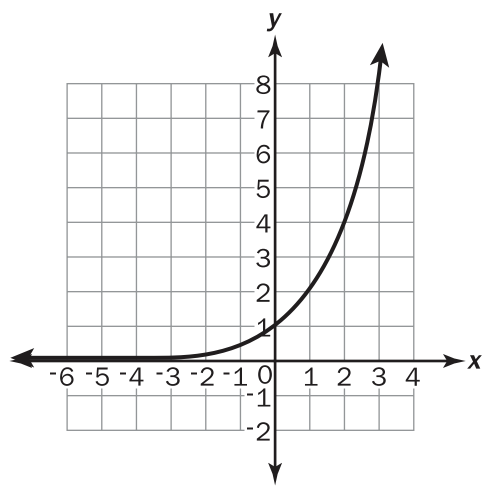
Algebra I Content Domain: Functions

Standard: MGSE9-12.F.IF.1. Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and $f(x)$ is the output (an element of the range). Graphically, the graph is $y = f(x)$.

Which of these is NOT a function?

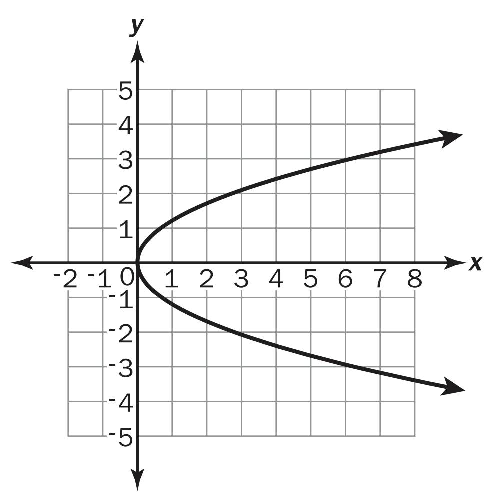
A. (5, 3), (6, 4), (7, 3), (8, 4)

B.



C. $y = 3x^2$

D.



Correct Answer: D

Explanation of Correct Answer: The correct answer is choice (D). A function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range, but for the graph of this parabola, there are two *y*-values for *x*; for example, at $x = 6$, y is 3 and –3. Therefore, it does not meet the definition of a function. Choices (A), (B), and (C) are functions because for every *x*-value, there is only one *y*-value.

Example Item 2

DOK Level 2: This is a DOK Level 2 item because it requires basic reasoning and asks students to apply their knowledge of functions that are undefined and extend that concept to determine the domain of this function.

Algebra I Content Domain: Functions

Standard: MGSE9-12.F.IF.1. Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e., each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and $f(x)$ is the output (an element of the range). Graphically, the graph is $y = f(x)$.

The number of school buses needed to transport students on a field trip is given by the function $f(x) = \dfrac{x + 3}{30}$. What is the domain of this function?

- **A.** x is the set of all real numbers.
- **B.** x is the set of all integers.
- **C.** x is the set of all non-negative integers.
- **D.** x is the set of all non-negative real numbers.

Correct Answer: C

Explanation of Correct Answer: The correct answer is choice (C), x is the set of all non-negative integers. Choices (A), (B), and (D) would include either fractional numbers, negative numbers, or both. The number of buses must be a positive and whole number.

Example Item 3

DOK Level 3: This is a DOK Level 3 item because it asks students for complex reasoning to generalize data from the table to create an equation to model the relationship between time and height of water. This equation can be used to make predictions and determine water level at any given time.

Algebra I Content Domain: Functions

Standard: MGSE9-12.F.LE.1b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

A partially filled container of water is being refilled from a garden hose at a constant rate. Erin records the height of the water at the end of different intervals of time in this table.

Elapsed Time (minutes)	Height of Water (cm)
3	43.1
7	53.9
9	59.3
15	75.5

Part A: What was the height of the water in the container before it was refilled? Round your answer to the nearest whole number.

Part B: After how many minutes did the height of the water double its original amount? Round your answer to the nearest whole minute.

Part C: Does this function model this relationship: $h = 2.7m + 35$? Explain why or why not.

Scoring Rubric

<table>
<tr><th>Points</th><th>Description</th></tr>
<tr><td>4</td><td>The response achieves the following:
Student demonstrates a complete and thorough understanding of recognizing situations in which one quantity changes at a constant rate per unit interval relative to another. Student also demonstrates a complete understanding of writing a function to model that situation. Award 4 points for a student response that contains all of the following elements:
Part A: 35 cm
Part B: 12.96 or about 13 minutes
Part C: Student states function is correct because it shows the correct rate of change AND the initial amount of water.</td></tr>
<tr><td>3</td><td>The response achieves the following:
Student demonstrates nearly complete understanding of recognizing situations in which one quantity changes at a constant rate per unit interval relative to another. Student also demonstrates a nearly complete understanding of writing a function to model that situation. Award 3 points for a student response that contains any 3 of the following elements:
Part A: 35 cm
Part B: 12.96 or about 13 minutes
Part C: Student states function is correct OR explains that the function shows the correct rate of change and the initial amount of water.</td></tr>
<tr><td>2</td><td>The response achieves the following:
Student demonstrates partial understanding of recognizing situations in which one quantity changes at a constant rate per unit interval relative to another. Student also demonstrates a partial understanding of writing a function to model that situation. Award 2 points for a student response that contains any 2 of the following elements:
Part A: 35 cm
Part B: 12.96 or about 13 minutes
Part C: Student states function is correct OR explains that the function shows the correct rate of change and the initial amount of water.</td></tr>
<tr><td>1</td><td>The response achieves the following:
Student demonstrates minimal understanding of recognizing situations in which one quantity changes at a constant rate per unit interval relative to another. Student also demonstrates a partial understanding of writing a function to model that situation. Award 1 point for a student response that contains any 1 of the following elements:
Part A: 35 cm
Part B: 12.96 or about 13 minutes
Part C: Student states function is correct OR explains that the function shows the correct rate of change and the initial amount of water.</td></tr>
<tr><td>0</td><td>The response achieves the following:
Student demonstrates limited to no understanding of interpreting the parameters in an exponential function in terms of a context.</td></tr>
</table>

Exemplar Response

Points Awarded	Sample Response
4	Part A: 35 cm Part B: 13 minutes Part C: Function is correct because it shows the correct rate of change and the initial amount of water.
3	Part A: 35 cm Part B: 13 minutes Part C: The function is correct.
2	Part A: 43 cm Part B: 13 minutes Part C: Function shows the correct rate of change and the initial amount of water.
1	Part A: 35 cm Part B: 19 minutes Part C: $h = 2.7m + 43.1$
0	Part A: 43 cm Part B: 19 minutes Part C: Function is incorrect.

DESCRIPTION OF TEST FORMAT AND ORGANIZATION

The Georgia Milestones Algebra I EOC assessment consists of a total of 73 items. You will be asked to respond to selected-response (multiple-choice), constructed-response, and extended constructed-response items.

The test will be given in two sections.

- You may have up to 85 minutes per section to complete Sections 1 and 2.
- The total estimated testing time for the Algebra I EOC assessment ranges from approximately 120 to 170 minutes. Total testing time describes the amount of time you have to complete the assessment. It does not take into account the time required for the test examiner to complete pre-administration and post-administration activities (such as reading the standardized directions to students).
- Sections 1 and 2 may be administered on the same day or across two consecutive days, based on the district's testing protocols for the EOC measures (in keeping with state guidance).
- During the Algebra I EOC assessment, a formula sheet will be available for you to use. Another feature of the Algebra I assessment is that you may use a graphing calculator in calculator-approved sections.

Effect on Course Grade

It is important that you take this course and the EOC assessment very seriously.

- For students in Grade 10 or above beginning with the 2011–2012 school year, the final grade in each course is calculated by weighing the course grade 85% and the EOC score 15%.
- For students in Grade 9 beginning with the 2011–2012 school year and later, the final grade in each course is calculated by weighing the course grade 80% and the EOC score 20%.
- A student must have a final grade of at least 70% to pass the course and to earn credit toward graduation.

PREPARING FOR THE ALGEBRA I EOC ASSESSMENT

STUDY SKILLS

As you prepare for this test, ask yourself the following questions:

- How would you describe yourself as a student?
- What are your study skills strengths and/or weaknesses?
- How do you typically prepare for a classroom test?
- What study methods do you find particularly helpful?
- What is an ideal study situation or environment for you?
- How would you describe your actual study environment?
- How can you change the way you study to make your study time more productive?

ORGANIZATION—OR TAKING CONTROL OF YOUR WORLD

- Establish a study area that has minimal distractions.
- Gather your materials in advance.
- Develop and implement your study plan.

ACTIVE PARTICIPATION

The most important element in your preparation is *you*. You and your actions are the key ingredient. Your active studying helps you stay alert and be more productive. In short, you need to interact with the course content. Here's how you do it.

- Carefully read the information and then DO something with it. Mark the important material with a highlighter, circle it with a pen, write notes on it, or summarize the information in your own words.
- Ask questions. As you study, questions often come into your mind. Write them down and actively seek the answers.
- Create sample test questions and answer them.
- Find a friend who is also planning to take the test and quiz each other.

TEST-TAKING STRATEGIES

Part of preparing for a test is having a set of strategies you can draw from. Include these strategies in your plan:

- Read and understand the directions completely. If you are not sure, ask a teacher.
- Read each question and all of the answer choices carefully.
- If you use scratch paper, make sure you copy your work to your test accurately.

- ✻ Underline the important parts of each task. Make sure that your answer goes on the answer sheet.
- ✻ Be aware of time. If a question is taking too much time, come back to it later.
- ✻ Answer all questions. Check your answers for accuracy. For constructed-response questions, do as much as you can. Remember, partially right responses will earn a partial score.
- ✻ Stay calm and do the best you can.

PREPARING FOR THE ALGEBRA I EOC ASSESSMENT

Read this guide to help prepare for the Algebra I EOC assessment.

The section of the guide titled “Content of the Algebra I EOC Assessment” provides a snapshot of the Algebra I course. In addition to reading this guide, do the following to prepare to take the assessment:

- Read your resources and other materials.
- Think about what you learned, ask yourself questions, and answer them.
- Read and become familiar with the way questions are asked on the assessment.
- Look at the sample answers for the constructed-response items to familiarize yourself with the elements of the exemplary responses. The rubrics will explain what is expected of you, point by point.
- Answer some practice Algebra I questions.
- There are additional items to practice your skills available online. Ask your teacher about online practice sites that are available for your use.

CONTENT OF THE ALGEBRA I EOC ASSESSMENT

Up to this point in the guide, you have been learning how to prepare for taking the EOC assessment. Now you will learn about the topics and standards that are assessed in the Algebra I EOC assessment and will see some sample items.

- The first part of this section focuses on what will be tested. It also includes sample items that will let you apply what you have learned in your classes and from this guide.
- The second part of this section contains additional items to practice your skills.
- The next part contains a table that shows the standard assessed for each item, the DOK level, the correct answer (key), and a rationale/explanation of the right and wrong answers for the additional practice items.
- You can use the sample items to familiarize yourself with the item formats found on the assessment.

  All example and sample items contained in this guide are the property of the Georgia Department of Education.

The Algebra I EOC assessment will assess the Algebra I standards documented at www.georgiastandards.org.

The content of the assessment is organized into three groupings, or domains, of standards for the purpose of providing feedback on student performance.

- A content domain is a reporting category that *broadly* describes and defines the content of the course, as measured by the EOC assessment.
- On the actual test the standards for Algebra I are grouped into three domains that follow your classwork: Algebra, Functions, and Algebra Connections to Statistics and Probability.
- Each domain was created by organizing standards that share similar content characteristics.
- The content standards describe the level of understanding each student is expected to achieve. They include the knowledge, concepts, and skills assessed on the EOC assessment, and they are used to plan instruction throughout the course.

SNAPSHOT OF THE COURSE

This section of the guide is organized into six units that review the material covered within the three domains of the Algebra I course. The material is presented by concept rather than by category or standard. In each unit you will find sample items similar to what you will see on the EOC assessment. The next section of the guide contains additional items to practice your skills followed by a table that shows for each item the standard assessed, the DOK level, the correct answer (key), and a rationale/explanation about the key and distractors.

All example and sample items contained in this guide are the property of the Georgia Department of Education.

The more you understand about the standards in each unit, the greater your chances of getting a good score on the EOC assessment.

UNIT 1: RELATIONSHIPS BETWEEN QUANTITIES AND EXPRESSIONS

In this unit, students study quantitative relationships. They rewrite expressions involving radicals (i.e., simplify and/or use the operations of addition, subtraction, and multiplication, with radicals within expressions limited to square roots). Students also interpret expressions and perform arithmetic operations (add, subtract, and multiply) on polynomials.

Use Properties of Rational and Irrational Numbers

MGSE9-12.N.RN.2 Rewrite expressions involving radicals.

MGSE9-12.N.RN.3 Explain why the sum or product of rational numbers is rational; why the sum of a rational number and an irrational number is irrational; and why the product of a nonzero rational number and an irrational number is irrational.

KEY IDEAS

1. A ***rational number*** is a real number that can be represented as a ratio $\frac{p}{q}$ such that p and q are both integers and $q \neq 0$. All rational numbers can be expressed as a decimal that stops or repeats.

   Examples:

   $-0.5, 0, 7, \frac{3}{2}, 0.2\overline{6}$

2. An ***irrational number*** is a real number that cannot be expressed as a ratio $\frac{p}{q}$ such that p and q are both integers and $q \neq 0$. Irrational numbers cannot be represented by decimals that stop or repeat.

Examples:

$\sqrt{3}$, π, $\frac{\sqrt{5}}{2}$

3. The sum of an irrational number and a rational number is always irrational. The product of a nonzero rational number and an irrational number is always irrational. The sum or product of rational numbers is rational.

Example:

The sum is irrational since it cannot be written as a fraction and the sum of a rational number and an irrational number is irrational.

Solution:

Let a be an irrational number, and let b be a rational number. Suppose that the sum of a and b is a rational number, c. If you can show that this is not true, it is the same as proving the original statement.

Let $b = \frac{p}{q}$, where p and q are integers and $q \neq 0$.

Let $c = \frac{m}{n}$, where m and n are integers and $n \neq 0$.

Substitute $\frac{p}{q}$ and $\frac{m}{n}$ for b and c. Then subtract to find a.

$$a + b = c$$

$$a + \frac{p}{q} = \frac{m}{n}$$

$$a = \frac{m}{n} - \frac{p}{q}$$

$$a = \frac{mq}{nq} - \frac{pn}{nq}$$

$$a = \frac{mq - pn}{nq}$$

The set of integers is closed under multiplication and subtraction, so $\frac{mq - pn}{nq}$ is an integer divided by an integer. This means that a is rational. However, a was assumed to be irrational, so this is a contradiction. This means that c must be irrational. So, the sum of an irrational number and a rational number is irrational.

4. Rewrite expressions involving radicals.

Examples:

$$3\sqrt{700}$$

$$3\sqrt{700 \bullet 100}$$

$$3 \bullet 10\sqrt{7}$$

$$30\sqrt{7}$$

Example:

Is the sum of 0.75 and –2.25 a rational or an irrational number?

Solution:

The sum is a rational number. The sum is –1.50, which can be rewritten as the fraction $\frac{-150}{100}$.

Example:

Is the sum of $\frac{1}{2}$ and $\sqrt{2}$ a rational or an irrational number?

Solution:

The sum is an irrational number. The square root of 2 is a decimal that does not terminate or repeat. Therefore, the actual sum can be written only as $\frac{1}{2} + \sqrt{2}$.

Example:

Is the product of –0.5 and $\sqrt{3}$ a rational or an irrational number? Explain your reasoning.

Solution:

The product is an irrational number. The square root of 3 is a decimal that does not terminate or repeat. Therefore, the product can be written only as $-0.5\sqrt{3}$.

REVIEW EXAMPLES

1. Rewrite.

$$\sqrt{2} \bullet \sqrt{72} \bullet \sqrt{5}$$

Solution:

$\sqrt{2 \bullet 72 \bullet 5} = \sqrt{144 \bullet 5} = \sqrt{12^2 \bullet 5} = 12\sqrt{5}$

Since 144 is a perfect square, the square root of 144 can be written as 12.

2. Is the sum of $\sqrt{3}$ and $\frac{1}{3}$ rational or irrational?

Solution:
The sum is irrational since it cannot be written as a fraction and the sum of a rational number and an irrational number is irrational.

3. Is the sum of $0.0\overline{675}$ and 8 rational or irrational?

Solution:
Since $0.0\overline{675}$ is repeating, it can be written as a fraction, $\frac{5}{74}$, so it is a rational number. The sum can be written as a fraction. Therefore, the sum is a rational number.

SAMPLE ITEMS

1. Look at the radical.

$$-8\sqrt{726}$$

What is a rewritten form of the radical?

A. $-88\sqrt{6}$
B. -90.75
C. $-986\sqrt{6}$
D. $-2{,}904$

Correct Answer: A

2. Look at the expression.

$$2\sqrt{8} \bullet \sqrt{20}$$

Which of these is equivalent to this expression?

A. $2\sqrt{28}$
B. 5
C. $8\sqrt{10}$
D. $32\sqrt{10}$

Correct Answer: C

3. Which sum is rational?

A. $\pi + 18$
B. $\sqrt{25} + 1.75$
C. $\sqrt{3} + 5.5$
D. $\pi + \sqrt{2}$

Correct Answer: B

4. Which product is irrational?

A. $\sqrt{2} \bullet \sqrt{50}$
B. $\sqrt{64} \bullet \sqrt{4}$
C. $\sqrt{9} \bullet \sqrt{49}$
D. $\sqrt{10} \bullet \sqrt{8}$

Correct Answer: D

Reason Quantitatively and Use Units to Solve Problems

MGSE9-12.N.Q.1 Use units of measure (linear, area, capacity, rates, and time) as a way to understand problems:

a. Identify, use, and record appropriate units of measure within context, within data displays, and on graphs;

b. Convert units and rates using dimensional analysis (English-to-English and Metric-to-Metric without conversion factor provided and between English and Metric with conversion factor);

c. Use units within multistep problems and formulas; interpret units of input and resulting units of output.

MGSE9-12.N.Q.2 Define appropriate quantities for the purpose of descriptive modeling. Given a situation, context, or problem, students will determine, identify, and use appropriate quantities for representing the situation.

MGSE9-12.N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. *For example, money situations are generally reported to the nearest cent (hundredth). Also, an answer's precision is limited to the precision of the data given.*

KEY IDEAS

1. A ***quantity*** is an exact amount or measurement. One type of quantity is a simple count, such as 5 eggs or 12 months. A second type of quantity is a measurement, which is an amount of a specific unit. Examples are 6 feet and 3 pounds.
2. A quantity can be exact or approximate. When an approximate quantity is used, it is important that we consider its level of accuracy. When working with measurements, we need to determine what level of accuracy is necessary and practical. For example, a dosage of medicine would need to be very precise. An example of a measurement that does not need to be very precise is the distance from your house to a local mall. The use of an appropriate unit for measurements is also important. For example, if you want to calculate the diameter of the Sun, you would want to choose a very large unit as your measure of length, such as miles or kilometers. Conversion of units can require approximations.

   Example:

   Convert 309 yards to feet.

   Solution:

   We know 1 yard is 3 feet.

   $$309 \text{ yards} \times \frac{3 \text{ feet}}{1 \text{ yard}} = 927 \text{ feet}$$

   There are 3 feet in 1 yard. This ratio can be written as a fraction. Since the multiplication contains yards in the numerator and denominator, yards will cancel. We can approximate that 309 yards is close to 900 feet.

3. The context of a problem tells us what types of units are involved. ***Dimensional analysis*** is a way to determine relationships among quantities using their dimensions, units, or unit equivalencies. Dimensional analysis suggests which quantities should be used for computation in order to obtain the desired result.

   Example:

   The cost, in dollars, of a single-story home can be approximated using the formula $C = klw$, where l is the approximate length of the home and w is the approximate width of the home. Find the units for the coefficient k.

   Solution:

   The coefficient k is a rate of cost, in dollars, for homes. To find the units for k, solve the equation $C = klw$, and then look at the units.

   $$C \text{ dollars} = k \times l \text{ feet} \times w \text{ feet}$$
   $$C = klw$$
   $$\frac{C}{lw} = k$$
   $$\frac{C \text{ dollars}}{lw \text{ feet}} \cdot \text{feet} = k$$

   The value of k is $\frac{C}{lw}$, and the unit is dollars per feet squared or dollars per square foot.

   You can check this using dimensional analysis:

   $$C = klw$$
   $$C = \frac{k \text{ dollars}}{\cancel{\text{feet}} \cdot \cancel{\text{feet}}} \cdot l\ \cancel{\text{feet}} \cdot k\ \cancel{\text{feet}}$$
   $$C = klw \text{ dollars}$$

4. The process of dimensional analysis is also used to convert from one unit to another. Knowing the relationship between units is essential for unit conversion.

   Example:

   Convert 45 miles per hour to feet per minute.

   Solution:

   To convert the given units, we use a form of dimensional analysis. We will multiply 45 mph by a series of ratios where the numerator and denominator are in different units but equivalent to each other. The ratios are carefully chosen to introduce the desired units.

   $$\frac{45 \text{ miles}}{\text{hr}} \times \frac{1 \text{ hr}}{60 \text{ minutes}} \times \frac{5{,}280 \text{ feet}}{1 \text{ mile}} = \frac{45 \times 5{,}280 \text{ feet}}{60 \times 1 \text{ minute}} = 3{,}960 \text{ feet per minute}$$

5. When data are displayed in a graph, the units and scale features are keys to interpreting the data. Breaks or an abbreviated scale in a graph should be noted as they can cause a misinterpretation of the data.

6. The measurements we use are often approximations. It is routinely necessary to determine reasonable approximations.

Example:

When Justin goes to work, he drives at an average speed of 65 miles per hour. It takes about 1 hour and 30 minutes for Justin to arrive at work. His car travels about 25 miles per gallon of gas. If gas costs \$3.65 per gallon, how much money does Justin spend on gas to travel to work?

Solution:

First, calculate the distance Justin travels.

$$65 \text{ miles per hour} \cdot 1.5 \text{ hour} = 97.5 \text{ miles}$$

Justin can travel 25 miles on 1 gallon of gas. Because 97.5 miles is close to 100 miles, he needs about $100 \div 25 = 4$ gallons of gas.

To find the cost of gas to travel to work, multiply cost per gallon by the number of gallons.

$$4 \times \$3.65 = \$14.60$$

Important Tips

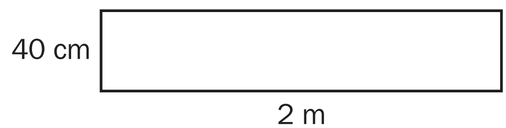
- ✍ When referring to a quantity, include the unit or the items being counted whenever possible.
- ✍ It is important to use appropriate units for measurements and to understand the relative sizes of units for the same measurement. You will need to know how to convert between units and how to round or limit the number of digits you use.
- ✍ Use units to help determine if your answer is reasonable. For example, if a question asks for a weight and you find an answer in feet, check your answer.

REVIEW EXAMPLES

1. The formula for density d is $d = \frac{m}{v}$, where m is mass and v is volume. If mass is measured in kilograms and volume is measured in cubic meters, what is the unit for density?

Solution:
The unit for density is $\frac{\text{kilograms}}{\text{meters}^3}$, or $\frac{\text{kg}}{\text{m}^3}$.

2. A rectangle has a length of 2 meters and a width of 40 centimeters. What is the perimeter of the rectangle?



Solution:
The perimeter of a rectangle could be found by using the formula $P = 2l + 2w$, where P is perimeter, l is length, and w is width. The perimeter could also be found by adding all side lengths.

To find the perimeter, both measurements need to have the same units. Convert 2 meters to centimeters or convert 40 centimeters to meters. Both methods are shown.

Method 1	Method 2
$2\text{m} \times \frac{100\text{cm}}{1\text{m}}$	$2\text{cm} \times \frac{1\text{m}}{100\text{cm}}$

Cancel the like units and multiply the remaining factors. The product is the converted measurement.

Method 1	Method 2
$2\text{m} \times \frac{100\text{cm}}{1\text{m}} = 200\text{cm}$	$2\text{cm} \times \frac{1\text{m}}{100\text{cm}} = 0.4\text{m}$

Use the converted measurement in the formula to find the perimeter.

Method 1	Method 2
$P = 2l + 2w$	$P = 2l + 2w$
$P = 2(200) + 2(40)$	$P = 2(2) + 2(0.4)$
$P = 400 + 80$	$P = 4 + 0.8$
$P = 480\text{cm}$	$P = 4.8\text{m}$

SAMPLE ITEMS

1. A rectangle has a length of 12 meters and a width of 400 centimeters. What is the perimeter, in cm, of the rectangle?

A. 824 cm
B. 1,600 cm
C. 2,000 cm
D. 3,200 cm

Correct Answer: D

2. Jill swam 200 meters in 2 minutes 42 seconds. If each lap is 50 meters long, which is MOST LIKELY to be her time, in seconds, per lap?

A. 32 seconds
B. 40 seconds
C. 48 seconds
D. 60 seconds

Correct Answer: B

Interpret the Structure of Expressions

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

KEY IDEAS

1. An ***algebraic expression*** contains variables, numbers, and operation symbols.
2. A ***term*** in an algebraic expression can be a constant, a variable, or a constant multiplied by a variable or variables. Every term is separated by a plus sign.

   Example:

   The terms in the expression $5x^2 - 3x + 8$ are $5x^2$, $-3x$, and 8.

3. A ***coefficient*** is the constant number that is multiplied by a variable in a term.

   Example:

   The coefficient in the term $7x^2$ is 7.

4. A ***common factor*** is a variable or number that terms can be divided by without a remainder. ***Factors*** are numbers multiplied together to get another number.

   Example:

   The common factors of $30x^2$ and $6x$ are 1, 2, 3, 6, and x.

5. A ***common factor of an expression*** is a number or term that the entire expression can be divided by without a remainder.

   Example:

   The common factor for the expression $3x^3 + 6x^2 - 15x$ is $3x$
   [(because $3x^3 + 6x^2 - 15x = 3x(x^2 + 2x - 5)$)].

6. If parts of an expression are independent of each other, the expression can be interpreted in different ways.

   Example:

   In the expression $\frac{1}{2}h(b_1 + b_2)$, the factors h and $(b_1 + b_2)$ are independent of each other. It can be interpreted as the product of h and a term that does not depend on h.

REVIEW EXAMPLES

1. Consider the expression $3n^2 + n + 2$.

   a. What is the coefficient of n?

   b. What terms are being added in the expression?

   Solution:

   a. 1

   b. $3n^2$, n, and 2

2. Look at one of the formulas for the perimeter of a rectangle where l represents the length and w represents the width.

$$2(l + w)$$

   What does the 2 represent in this formula?

   Solution:
   The 2 represents the two sets of length/width pairs.

SAMPLE ITEMS

1. In which expression is the coefficient of term "*n*" –1?

A. $3n^2 + 4n - 1$
B. $-n^2 + 5n + 4$
C. $-2n^2 - n + 5$
D. $4n^2 + n - 5$

Correct Answer: C

2. The expression s^2 is used to calculate the area of a square, where *s* is the side length of the square. What does the expression $(8x)^2$ represent?

A. the area of a square with a side length of 8
B. the area of a square with a side length of 16
C. the area of a square with a side length of $4x$
D. the area of a square with a side length of $8x$

Correct Answer: D

Perform Arithmetic Operations on Polynomials

MGSE9-12.A.APR.1 Add, subtract, and multiply polynomials; understand that polynomials form a system analogous to the integers in that they are closed under these operations.

KEY IDEAS

1. A ***polynomial*** is an expression made from one or more terms that involve constants, variables, and exponents.

   Examples:

   $$3x$$

   $$x^3 + 5x^2 + 4$$

   $$a^2b - 2ab + b^2$$

2. To add and subtract polynomials, combine like terms. In a polynomial, like terms have the same variables and are raised to the same powers.

   Examples:

   $$\begin{aligned} 7x + 6 + 5x - 3 &= 7x + 5x + 6 - 3 \\ &= 12x + 3 \end{aligned}$$

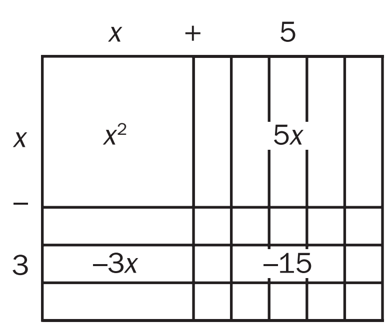
   $$\begin{aligned} 13a + 1 - (5a - 4) &= 13a + 1 - 5a + 4 \\ &= 13a - 5a + 1 + 4 \\ &= 8a + 5 \end{aligned}$$

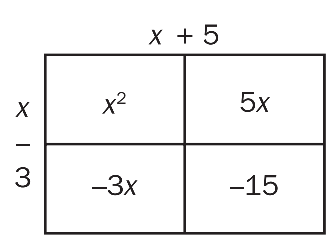
3. To multiply polynomials, use the Distributive Property. Multiply every term in the first polynomial by every term in the second polynomial. To completely simplify, add like terms after multiplying.

   Example:

   $$\begin{aligned} (x + 5)(x - 3) &= (x)(x) + (-3)(x) + (5)(x) + (5)(-3) \\ &= x^2 - 3x + 5x - 15 \\ &= x^2 + 2x - 15 \end{aligned}$$

   The multiplication can also be represented with tiles and area models.

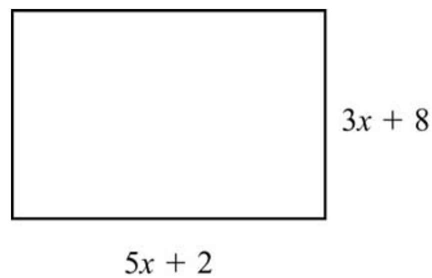




4. Polynomials are closed under addition, subtraction, and multiplication, similar to the set of integers. This means that the sum, difference, or product of two polynomials is always a polynomial.

REVIEW EXAMPLES

1. The dimensions of a rectangle are shown.



What is the perimeter, in units, of the rectangle?

Solution:

Substitute $5x + 2$ for l and $3x + 8$ for w in the formula for the perimeter of a rectangle:

$$P = l + l + w + w$$
$$P = 2l + 2w$$
$$P = 2(5x + 2) + 2(3x + 8)$$
$$P = 10x + 4 + 6x + 16$$
$$P = 10x + 6x + 4 + 16$$
$$P = 16x + 20 \text{ units}$$

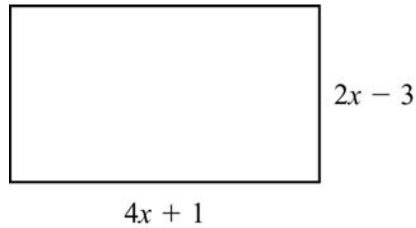
2. Rewrite the expression $(x^3 + 2x^2 - x) - (-x^3 + 2x^2 + 6)$.

Solution:

Combine like terms:

$$\begin{aligned}(x^3 + 2x^2 - x) - (-x^3 + 2x^2 + 6) &= x^3 + 2x^2 - x + x^3 - 2x^2 - 6\\ &= x^3 + x^3 + 2x^2 - 2x^2 - x - 6\\ &= 2x^3 - x - 6\end{aligned}$$

3. The dimensions of a patio, in feet, are shown below.



What is the area of the patio, in square feet?

Solution:

Substitute $4x + 1$ for b and $2x - 3$ for h in the formula for the area of a rectangle:

$$A = bh$$
$$A = (4x + 1)(2x - 3)$$
$$A = 8x^2 - 12x + 2x - 3$$
$$A = 8x^2 - 10x - 3 \text{ square feet}$$

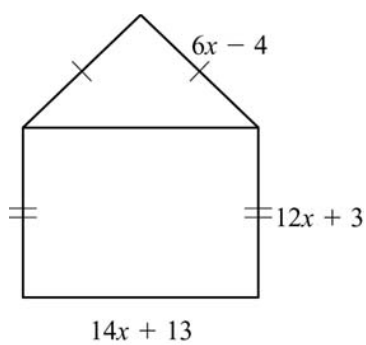
SAMPLE ITEMS

1. What is the product of $7x - 4$ and $8x + 5$?

A. $15x + 1$
B. $30x + 2$
C. $56x^2 + 3x - 20$
D. $56x^2 - 3x + 20$

Correct Answer: C

2. A model of a house is shown.



What is the perimeter, in units, of the model?

A. $32x + 12$ units
B. $46x + 25$ units
C. $50x + 11$ units
D. $64x + 24$ units

Correct Answer: C

3. Which expression has the same value as the expression $(8x^2 + 2x - 6) - (5x^2 - 3x + 2)$?

A. $3x^2 - x - 4$
B. $3x^2 + 5x - 8$
C. $13x^2 - x - 8$
D. $13x^2 - 5x - 4$

Correct Answer: B

UNIT 2: REASONING WITH LINEAR EQUATIONS AND INEQUALITIES

This unit investigates linear equations and inequalities. Students create linear equations and inequalities and use them to solve problems. They learn the process of reasoning and justify the steps used to solve simple equations. Students also solve systems of equations and represent linear equations and inequalities graphically. They write linear functions that describe a relationship between two quantities and write arithmetic sequences recursively and explicitly. They understand the concept of a function and use function notation. Given tables, graphs, and verbal descriptions, students interpret key characteristics of linear functions and analyze these functions using different representations.

Solving Equations and Inequalities in One Variable

MGSE9-12.A.REI.3 Solve linear equations and inequalities in one variable including equations with coefficients represented by letters. *For example, given ax + 3 = 7, solve for x.*
Solve systems of equations.

MGSE9-12.A.REI.5 Show and explain why the elimination method works to solve a system of two-variable equations.

MGSE9-12.A.REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

KEY IDEAS

1. Solving an equation or inequality means finding the quantity or quantities that make the equation or inequality true. The strategies for solving an equation or inequality depend on the number of variables and the exponents that are included.
2. Here is an algebraic method for solving a linear equation with one variable:

   Apply algebraic properties to write equivalent expressions until the desired variable is isolated on one side. Be sure to check your answers.

   Example:

   Solve $2(3 - a) = 18$.

   Solution:

   Solve the equation using either of these two ways:

$2(3 - a) = 18$	$2(3 - a) = 18$	
$3 - a = 9$	$6 - 2a = 18$	Distributive Property
$-a = 6$	$-2a = 12$	Addition Property of Equality
$a = -6$	$a = -6$	Multiplicative Inverses and Identity Property of 1

3. Here is an algebraic method for solving a linear inequality with one variable:

   Write equivalent expressions until the desired variable is isolated on one side. If you multiply or divide by a negative number, make sure you reverse the inequality symbol.

   Example:

   Solve $2(5 - x) > 8$ for x.

   Solution:

   Solve the inequality using either of these two ways:

$2(5-x)>8$	$2(5-x)>8$	
$5-x>4$	$10-2x>8$	Distributive Property
$-x>-1$	$-2x>-2$	Addition Property of Inequality
$x<1$	$x<1$	Multiplicative Inverses and Identity Property of 1

Important Tips

✍ If you multiply or divide both sides of an inequality by a negative number, make sure you reverse the inequality sign.

✍ Be familiar with the properties of equality and inequality so you can transform equations or inequalities.

- The ***addition property*** of equality tells us that adding the same number to each side of an equation gives us an equivalent equation.

  Example: if $a - b = c$, then $a - b + b = c + b$, or $a = c + b$

- The ***multiplication property*** of equality tells us that multiplying the same number to each side of an equation gives us an equivalent equation.

  Example: if $\frac{a}{b} = c$, then $\frac{a}{b} \cdot b = c \cdot b$, or $a = c \cdot b$

- The ***multiplication inverse property*** tells us that multiplying a number by its reciprocal equals 1.

  Example: $\frac{1}{a}(a) = 1$

- The ***additive inverse property*** tells us that adding a number to its opposite equals 0.

  Example: $a + (-a) = 0$

✍ Sometimes eliminating denominators by multiplying all terms by a common denominator or common multiple makes it easier to solve an equation or inequality.

REVIEW EXAMPLES

1. Karla wants to save up for a prom dress. She figures she can save \$9 each week from the money she earns babysitting. If she plans to spend less than \$150 for the dress, how many weeks will it take her to save enough money to buy any dress in her price range?

   Solution:

   Let w represent the number of weeks. If she saves \$9 each week, Karla will save $9w$ dollars after w weeks. We need to determine the minimum number of weeks it will take her to save \$150. Use the inequality $9w \geq 150$ to solve the problem. We need to transform $9w \geq 150$ to isolate w. Divide both sides by 9 to get $w \geq 16\frac{2}{3}$ weeks. Because we do not know what day Karla gets paid each week, we need the answer to be a whole number. So, the answer has to be 17, the smallest whole number greater than $16\frac{2}{3}$. She will save \$144 after 16 weeks and \$153 after 17 weeks.

2. Joachim wants to know if he can afford to add texting to his cell phone plan. He currently spends \$21.49 per month for his cell phone plan, and the most he can spend for his cell phone is \$30 per month. He could get unlimited text messaging added to his plan for an additional \$10 each month. Or, he could get a "pay-as-you-go" plan that charges a flat rate of \$0.15 per text message. He assumes that he will send an average of 5 text messages per day. Can Joachim afford to add a text message plan to his cell phone plan?

   Solution:

   Joachim cannot afford either plan.

   At an additional \$10 per month for unlimited text messaging, Joachim's cell phone bill would be \$31.49 a month. If he chose the pay-as-you-go plan, each day he would expect to pay for 5 text messages. Let t stand for the number of text messages per month. Then, on the pay-as-you-go plan, Joachim could expect his cost to be represented by the expression:

   $$21.49 + 0.15t$$

   If he must keep his costs at \$30 or less, $21.49 + 0.15t \leq 30$.

   To find the number of text messages he can afford, solve for t.

$21.49 - 21.49 + 0.15t \leq 30 - 21.49$	Subtract 21.49 from both sides.
$0.15t \leq 8.51$	Combine like terms.
$t \leq 56.733\ldots$	Divide both sides by 0.15.

   The transformed inequality tells us that Joachim would need to send fewer than 57 text messages per month to afford the pay-as-you-go plan. However, 5 text messages per day at a minimum of 28 days in a month is 140 text messages per month. So, Joachim cannot afford text messages for a full month, and neither plan fits his budget.

3. Two cars start at the same point and travel in opposite directions. The first car travels 15 miles per hour faster than the second car. In 4 hours, the cars are 300 miles apart. Use the formula below to determine the rate of the second car.

$$4(r + 15) + 4r = 300$$

What is the rate, r, of the second car?

Solution:

The second car is traveling 30 miles per hour.

$4(r + 15) + 4r = 300$	Write the original equation.
$4r + 60 + 4r = 300$	Multiply 4 by $r + 15$.
$8r + 60 = 300$	Combine like terms.
$8r = 240$	Subtract 60 from each side.
$r = 30$	Divide each side by 8.

4. Solve the equation $14 = ax + 6$ for x. Show and justify your steps.

Solution:

$14 = ax + 6$	Write the original equation.
$14 - 6 = ax + 6 - 6$	Subtraction Property of Equality
$8 = ax$	Combine like terms on each side.
$\frac{8}{a} = \frac{ax}{a}$	Division Property of Equality; $a \neq 0$
$\frac{8}{a} = x$	Simplify.

SAMPLE ITEMS

1. This equation can be used to find *h*, the number of hours it will take Flo and Bryan to mow their lawn.

$$\frac{h}{3} + \frac{h}{6} = 1$$

How many hours will it take them to mow their lawn?

A. 6 hours
B. 3 hours
C. 2 hours
D. 1 hour

Correct Answer: C

2. A ferry boat carries passengers back and forth between two communities on the Peachville River.

- **It takes 30 minutes longer for the ferry to make the trip upstream than downstream.**
- **The ferry's average speed in still water is 15 miles per hour.**
- **The river's current is usually 5 miles per hour.**

This equation can be used to determine how many miles apart the two communities are.

$$\frac{m}{15 - 5} = \frac{m}{15 + 5} + 0.5$$

What is *m*, the distance between the two communities?

A. 0.5 mile
B. 5 miles
C. 10 miles
D. 15 miles

Correct Answer: C

3. For what values of x is the inequality $\frac{2}{3} + \frac{x}{3} > 1$ true?

A. $x < 1$
B. $x > 1$
C. $x < 5$
D. $x > 5$

Correct Answer: B

4. Look at the steps used when solving $3(x - 2) = 3$ for x.

$3(x - 2) = 3$	Write the original equation.
$3x - 6 = 3$	Use the Distributive Property.
$3x - 6 + 6 = 3 + 6$	Step 1
$3x = 9$	Step 2
$\frac{3x}{3} = \frac{9}{3}$	Step 3
$x = 3$	Step 4

Which step is the result of combining like terms?

A. Step 1
B. Step 2
C. Step 3
D. Step 4

Correct Answer: B

Solving a System of Two Linear Equations

MGSE9-12.A.REI.5 Show and explain why the elimination method works to solve a system of two-variable equations.

MGSE9-12.A.REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

KEY IDEAS

1. A system of linear equations consists of two or more linear equations that may or may not have a common solution. The solution of a system of two linear equations is the set of values for the variables that makes all the equations true. The solutions can be expressed as ordered pairs (x, y) or as two equations, one for x and the other for y (x = . . . and y = . . .).

   Strategies:

   * Use tables or graphs as strategies for solving a system of equations. For tables, use the same values for both equations. For graphs, the intersection of the graph of both equations provides the solution to the system of equations.

   Example:

   Solve this system of equations.

$$\begin{cases} y = 2x - 4 \\ x = y + 1 \end{cases}$$

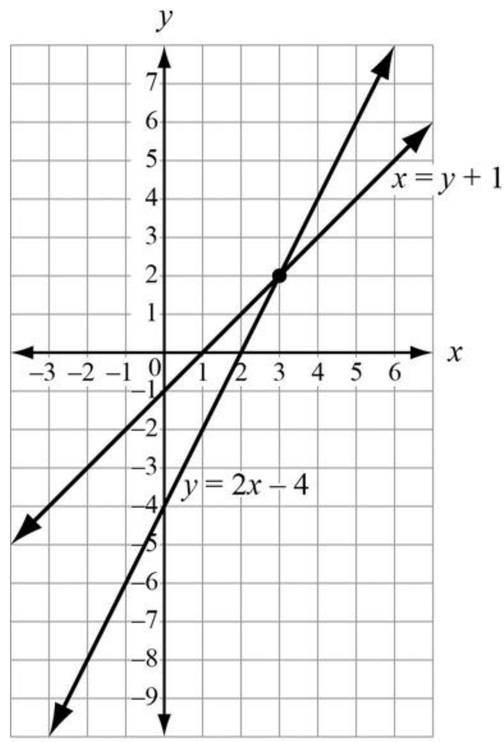
   Solution:

   First, find coordinates of points for each equation. Making a table of values for each is one way to do this. Use the same values for both equations.

x	$2x - 4$
–1	–6
0	–4
1	–2
2	0
3	2
4	4

x	y
–1	–2
0	–1
1	0
2	1
3	2
4	3

Graph the first equation by using the y-intercept, (0, −4), and the slope, 2. Graph the second equation by solving for y to get $y = x - 1$ and then use the y-intercept, (0, −1), and the slope, 1. Both equations are displayed on the graph below.



The graph shows all the ordered pairs of numbers (rows from the table), that satisfy $y = 2x - 4$ and the ordered pairs that satisfy $x = y + 1$. From the graph, it appears that the lines cross at about (3, 2). Then try that combination in both equations to determine whether (3, 2) is a solution to both equations.

$y = 2x - 4$	$x = y + 1$
$(2) = 2(3) - 4$	$3 = 2 + 1$
$2 = 6 - 4$	$3 = 3$
$2 = 2$	

So, (3, 2) is the solution to the system of equations. The graph also suggests that (3, 2) is the only point the lines have in common, so we have found the only pair of numbers that works for both equations.

Strategies:

* Simplify the problem by eliminating one of the two variables.

Substitution method: Use one equation to isolate a variable and replace that variable in the other equation with the equivalent expression you just found. Solve for the one remaining variable. Use the solution to the remaining variable to find the unknown you eliminated.

Example:

Solve this system of equations.

$$\begin{cases} 2x - y = 1 \\ 5 - 3x = 2y \end{cases}$$

Solution:

Begin by choosing one of the equations and solving for one of the variables. This variable is the one you will eliminate. We could solve the top equation for *y*.

$$\begin{aligned} 2x - y &= 1 \\ 2x &= 1 + y \\ 2x - 1 &= y \\ y &= 2x - 1 \end{aligned}$$

Next, use substitution to replace the variable you are eliminating in the other equation.

$$\begin{aligned} 5 - 3x &= 2y \\ 5 - 3x &= 2(2x - 1) \\ 5 - 3x &= 4x - 2 \\ 7 &= 7x \\ 1 &= x \end{aligned}$$

Now, find the corresponding *y*-value. You can use either equation.

$$\begin{aligned} 2x - y &= 1 \\ 2(1) - y &= 1 \\ 2 - y &= 1 \\ -y &= 1 - 2 \\ -y &= -1 \\ y &= 1 \end{aligned}$$

So, the solution is $x = 1$ and $y = 1$, or (1, 1).

Check solution:

$$\begin{aligned} 2x - y &= 1 \qquad & 5 - 3x &= 2y \\ 2(1) - (1) &= 1 \qquad & 5 - 3(1) &= 2(1) \\ 2 - 1 &= 1 \qquad & 5 - 3 &= 2 \\ 1 &= 1 \qquad & 2 &= 2 \end{aligned}$$

<u>Elimination method</u>: Add the equations (or a transformation of the equations) to eliminate a variable. Then solve for the remaining variable and use this value to find the value of the variable you eliminated.

Example:

Solve this system of equations.

$$\begin{cases} 2x - y = 1 \\ 5 - 3x = -y \end{cases}$$

Solution:

First, rewrite the second equation in standard form.

$$\begin{cases} 2x - y = 1 \\ -3x + y = -5 \end{cases}$$

Decide which variable to eliminate. We can eliminate the *y*-terms because they are opposites.

$$\begin{aligned} 2x - y &= 1 \\ -3x + y &= -5 \end{aligned}$$

Add the equations, term by term, eliminating *y* and reducing to one equation. This is an application of the addition property of equality.

$$-x = -4$$

Multiply both sides by –1 to solve for *x*. This is an application of the multiplication property of equality.

$$\begin{aligned} (-1)(-x) &= (-1)(-4) \\ x &= 4 \end{aligned}$$

Now substitute this value of *x* in either original equation to find *y*.

$$\begin{aligned} 2x - y &= 1 \\ 2(4) - y &= 1 \\ 8 - y &= 1 \\ -y &= -7 \\ y &= 7 \end{aligned}$$

The solution to the system of equations is (4, 7).

Check solution:

$$\begin{aligned} 2x - y &= 1 & \qquad -3x + y &= -5 \\ 2(4) - (7) &= 1 & \qquad -3(4) + (7) &= -5 \\ 8 - 7 &= 1 & \qquad -12 + 7 &= -5 \\ 1 &= 1 & \qquad -5 &= -5 \end{aligned}$$

Example:

Solve this system of equations.

$$\begin{cases} 3x - 2y = 7 \\ 2x - 3y = 3 \end{cases}$$

Solution:

First, decide which variable to eliminate. We can eliminate the *y*-terms but will need to change the coefficients of the *y*-terms so that they are the same value or opposites. We can multiply the first equation by 3 and the second equation by –2. This is the multiplication property of equality.

$$(3)(3x - 2y = 7) \rightarrow 9x - 6y = 21$$
$$(2)(2x - 3y = 3) \rightarrow 4x - 6y = 6$$

Subtract the second equation from the first equation, term by term, eliminating *y* and reducing to one equation. This is the addition property of equality.

$$5x = 15$$

Divide both sides by 5 to solve for *x*.

$$\frac{5x}{5} = \frac{15}{5}$$
$$x = 3$$

Now substitute this value of *x* in either original equation to find *y*.

$$\begin{aligned} 3x - 2y &= 7 \\ 3(3) - 2y &= 7 \\ 9 - 2y &= 7 \\ -2y &= -2 \\ y &= 1 \end{aligned}$$

The solution to the system of equations is (3, 1).

Check solution:

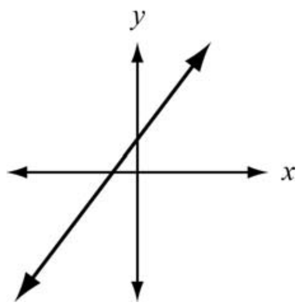
$$\begin{aligned} 3x - 2y &= 7 \\ 3(3) - 2(1) &= 7 \\ 9 - 2 &= 7 \\ 7 &= 7 \end{aligned} \qquad \begin{aligned} 2x - 3y &= 3 \\ 2(3) - 3(1) &= 3 \\ 6 - 3 &= 3 \\ 3 &= 3 \end{aligned}$$

2. The graphing method only suggests the solution of a system of equations. To check the solution, substitute the values into the equations and make sure the ordered pair satisfies both equations.
3. When graphing a system of equations:

    a. If the lines are parallel, then there is no solution to the system.

    b. If the lines coincide, then the lines have all their points in common and any pair of points that satisfies one equation will satisfy the other.

4. When using elimination to solve a system of equations, if both variables are removed when you try to eliminate one, and if the result is a true equation such as 0 = 0, then the lines coincide. The equations would have all ordered pairs in common, as shown in the following graph.



Example:

Solve this system of equations.

$$\begin{cases} 3x - 3y = 3 \\ x - y = 1 \end{cases}$$

Solution:

First, decide which variable to eliminate. We can eliminate the *y*-terms but will need to change the coefficients of the *y*-terms so that they are the same value or opposites. We can multiply the second equation by 3. This is the multiplication property of equality.

$$(3x - 3y = 3) \rightarrow 3x - 3y = 3$$
$$(3)(x - y = 1) \rightarrow 3x - 3y = 3$$

Subtract the second equation from the first equation, term by term, eliminating *y* and reducing to one equation. This is the addition property of equality.

$$0 = 0$$

The solution to the system of equations is any value of *x* that gives the same value of *y* for either equation.

Check solution:

Substitute $x = 1$ in either original equation to find *y*.

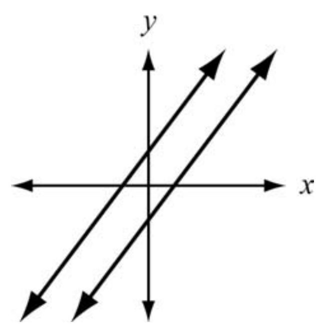
$$\begin{aligned} 3x - 3y &= 3 \\ 3(1) - 3y &= 3 \\ 3 - 3y &= 3 \\ -3y &= 0 \\ y &= 0 \end{aligned}$$

A solution to the system of equations is (1, 0).

Check solution:

$$\begin{aligned} 3x - 3y &= 3 \\ 3(1) - 3(0) &= 3 \\ 3 - 0 &= 3 \\ 3 &= 3 \end{aligned} \qquad \begin{aligned} x - y &= 1 \\ (1) - (0) &= 1 \\ 1 - 0 &= 1 \\ 1 &= 1 \end{aligned}$$

When using elimination to solve a system of equations, if the result is a false equation such as 3 = 7, then the lines are parallel. The system of equations has no solution since there is no point where the lines intersect.



Example:

Solve this system of equations.

$$\begin{cases} 3x - 3y = 7 \\ x - y = 1 \end{cases}$$

Solution:

First, decide which variable to eliminate. We can eliminate the *y*-terms but will need to change the coefficients of the *y*-terms so that they are the same value or opposites. We can multiply the second equation by 3. This is the multiplication property of equality.

$$(3x - 3y = 7) \rightarrow 3x - 3y = 7$$

$$(3)(x - y = 1) \rightarrow 3x - 3y = 3$$

Subtract the second equation from the first equation, term by term, eliminating *y* and reducing to one equation. This is the addition property of equality.

$$0 = 4$$

The system of equations has no solutions.

REVIEW EXAMPLES

1. Consider the equations $y = 2x - 3$ and $y = -x + 6$.

   a. Complete the tables below.

$y = 2x - 3$	
x	y

$y = -x + 6$	
x	y

   b. Is there an ordered pair that satisfies both equations? If so, what is it?

   c. Graph both equations on the same coordinate plane by plotting the ordered pairs from the tables and connecting the points.

   d. Do the lines appear to intersect? If so, where? How can you tell that the point where the lines appear to intersect is a common point for both lines?

Solution:

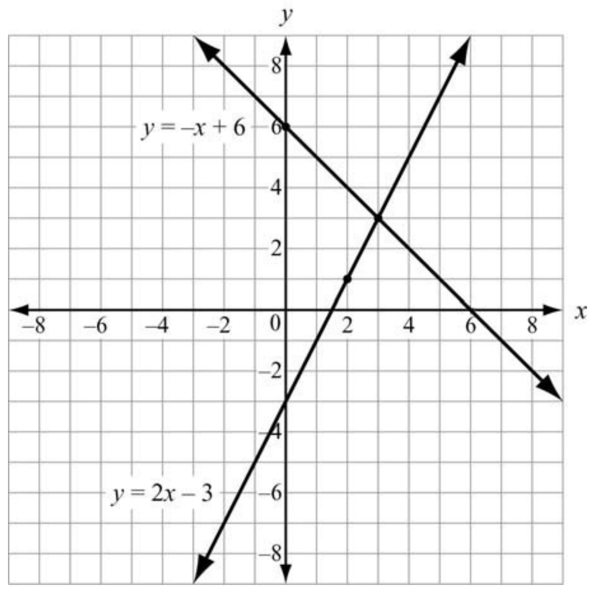
a.

$y = 2x - 3$	
x	y
–1	–5
0	–3
1	–1
2	1
3	3

$y = -x + 6$	
x	y
–1	7
0	6
1	5
2	4
3	3

b. Yes, the ordered pair (3, 3) satisfies both equations.

c.



d. The lines appear to intersect at (3, 3). When $x = 3$ and $y = 3$ are substituted into each equation, the values satisfy both equations. This proves that (3, 3) lies on both lines, which means it is a common solution to both equations.

2. Rebecca has five coins worth 65 cents in her pocket. If she only has quarters and nickels, how many quarters does she have? Use a system of equations to arrive at your answer and show all steps.

Solution:

If q represents the number of quarters and n represents the number of nickels, the two equations could be $25q + 5n = 65$ (value of quarters plus value of nickels is 65 cents) and $q + n = 5$ (she has 5 coins). The equations in the system would be $25q + 5n = 65$ and $q + n = 5$.

Next, solve $q + n = 5$ for q. By subtracting n from both sides, the result is $q = 5 - n$.

Next, eliminate q by replacing q with $5 - n$ in the other equation: $25(5 - n) + 5n = 65$.

Solve this equation for n.

$$25(5 - n) + 5n = 65$$

$$125 - 25n + 5n = 65$$

$$125 - 20n = 65$$

$$-20n = -60$$

$$n = 3$$

Now solve for q by replacing n with 3 in the equation $q = 5 - n$. So, $q = 5 - 3 = 2$, so 2 is the number of quarters.

Rebecca has 2 quarters and 3 nickels.

Check solution:

$25q + 5n = 65$	$q + n = 5$
$25(2) + 5(3) = 65$	$(2) + (3) = 5$
$50 + 15 = 65$	$2 + 3 = 5$
$65 = 65$	$5 = 5$

Strategies:

* An alternate method for finding the equations is to set up the equations in terms of dollars: $0.25q + 0.05n = 0.65$ and $q + n = 5$.

3. Peg and Larry purchased "no contract" cell phones. Peg's phone costs \$25 plus \$0.25 per minute. Larry's phone costs \$35 plus \$0.20 per minute. After how many minutes of use will Peg's phone cost more than Larry's phone?

Solution:

Let x represent the number of minutes used. Peg's phone costs $25 + 0.25x$. Larry's phone costs $35 + 0.20x$. We want Peg's cost to exceed Larry's.

This gives us $25 + 0.25x > 35 + 0.20x$, which we then solve for x.

$$25 + 0.25x > 35 + 0.20x$$
$$25 + 0.25x - 0.20x > 35 + 0.20x - 0.20x$$
$$25 + 0.05x > 35$$
$$25 - 25 + 0.05x > 35 - 25$$
$$0.05x > 10$$
$$\frac{0.05x}{0.05} > \frac{10}{0.05}$$
$$x > 200$$

After 200 minutes of use, Peg's phone will cost more than Larry's phone.

Check solution:

Since Peg's phone will cost more than Larry's phone after 200 minutes, we can substitute 201 minutes to check if it is true.

$$25 + 0.25x > 35 + 0.20x$$
$$25 + 0.25(201) > 35 + 0.20(201)$$
$$25 + 50.25 > 35 + 40.2$$
$$75.25 > 75.20$$

4. Is (3, –1) a solution of this system?

$$\begin{cases} y = 2 - x \\ 3 - 2y = 2x \end{cases}$$

Solution:

Substitute the coordinates (3, –1) into each equation.

$$\begin{aligned} y &= 2 - x \\ -1 &= 2 - 3 \\ -1 &= -1 \end{aligned} \qquad\qquad \begin{aligned} 3 - 2y &= 2x \\ 3 - 2(-1) &= 2(3) \\ 3 + 2 &= 6 \\ 5 &= 6 \end{aligned}$$

The coordinates of the given point do not satisfy $3 - 2y = 2x$. If you get a false equation when trying to solve a system algebraically, then it means that the coordinates of the point are not the solution. So, (3, –1) is not a solution of the system.

5. Solve this system.

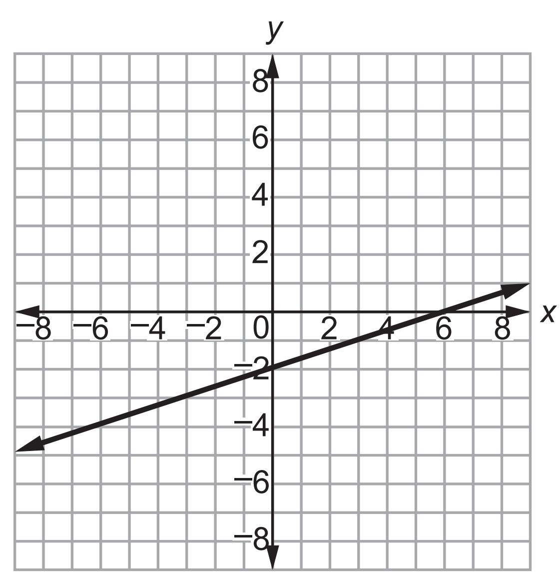
$$\begin{cases} x - 3y = 6 \\ -x + 3y = -6 \end{cases}$$

Solution:

Add the terms of the equations. Each pair of terms consists of opposites, and the result is 0 + 0 = 0.

$$\begin{array}{r} x - 3y = 6 \\ -x + 3y = -6 \\ \hline 0 = 0 \end{array}$$

This result is always true, so the two equations represent the same line. Every point on the line is a solution to the system.



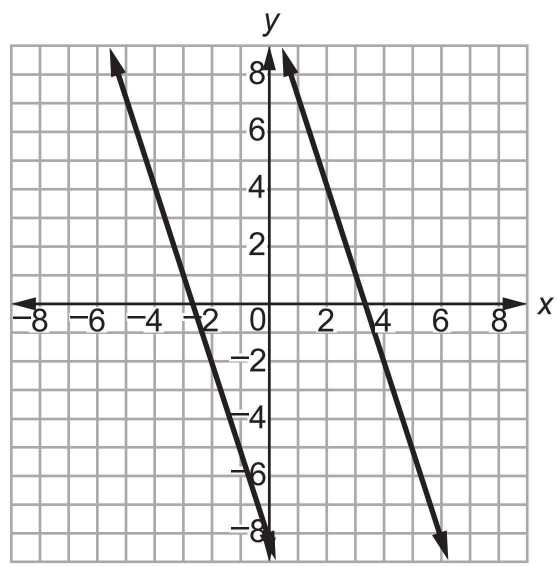
6. Solve this system.

$$\begin{cases} -3x - y = 10 \\ 3x + y = -8 \end{cases}$$

Solution:

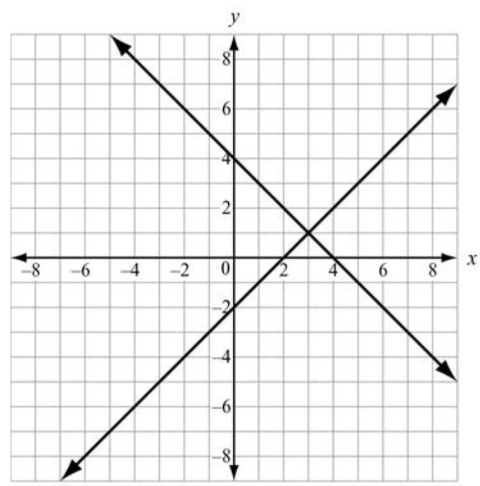
Add the terms in the equations: 0 = 2.

The result is never true. The two equations represent parallel lines. As a result, the system has no solution.



SAMPLE ITEMS

1. Two lines are graphed on this coordinate plane.



Which point appears to be a solution of the equations of both lines?

A. (0, –2)
B. (0, 4)
C. (2, 0)
D. (3, 1)

Correct Answer: D

2. Based on the tables, at what point do the lines $y = -x + 5$ and $y = 2x - 1$ intersect?

$y = -x + 5$	
x	y
–1	6
0	5
1	4
2	3
3	2

$y = 2x - 1$	
x	y
–1	–3
0	–1
1	1
2	3
3	5

A. (1, 1)
B. (3, 5)
C. (2, 3)
D. (3, 2)

Correct Answer: C

3. Look at the tables of values for two linear functions, $f(x)$ and $g(x)$.

x	$f(x)$
–1	16
0	7
1	4
3	–2
5	–8
7	–14

x	$g(x)$
–1	–18
0	–14
1	–10
3	–2
5	6
7	14

What is the solution to $f(x) = g(x)$?

Solution:

The solution to $f(x) = g(x)$ is $x = 3$. This is the value of x where $f(x)$ and $g(x)$ both equal –2.

4. Which ordered pair is a solution of $3y + 2 = 2x - 5$?

A. (–5, 2)
B. (0, –5)
C. (5, 1)
D. (7, 5)

Explanation of correct answer: The correct answer is choice (C) (5, 1). Also, if the values of x and y are substituted into the equation, the statement becomes 5 = 5, which is a true statement. This shows that the ordered pair is a solution of the equation.

Correct Answer: C

5. A manager is comparing the cost of buying baseball caps from two different companies.

- **Company X charges a $50 fee plus $7 per baseball cap.**
- **Company Y charges a $30 fee plus $9 per baseball cap.**

For what number of baseball caps will the cost be the same at both companies?

A. 10
B. 20
C. 40
D. 100

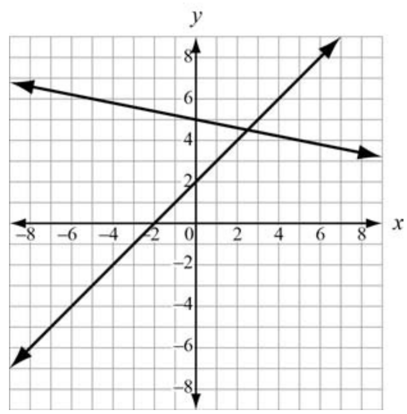
Correct Answer: A

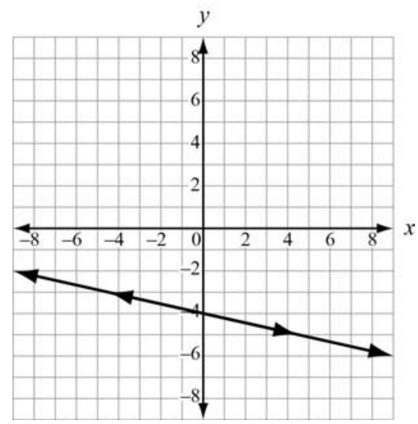
6. A shop sells one-pound bags of peanuts for $2 and three-pound bags of peanuts for $5. If 9 bags are purchased for a total cost of $36, how many three-pound bags were purchased?

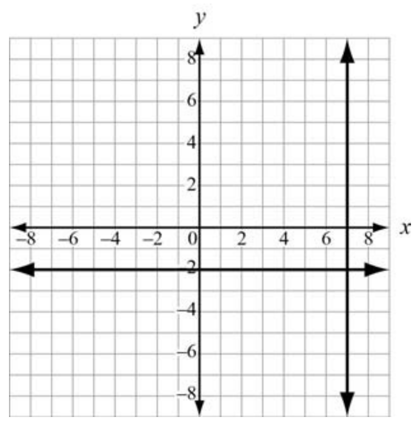
A. 3
B. 6
C. 9
D. 18

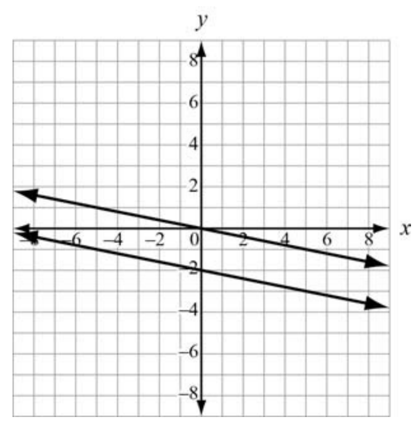
Correct Answer: B

7. Which graph represents a system of linear equations that has multiple common coordinate pairs?

A. 

B. 

C. 

D. 

Correct Answer: B

Represent and Solve Equations and Inequalities Graphically

MGSE9-12.A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane.

MGSE9-12.A.REI.11 Using graphs, tables, or successive approximations, show that the solution to the equation $f(x) = g(x)$ is the x-value where the y-values of $f(x)$ and $g(x)$ are the same.

MGSE9-12.A.REI.12 Graph the solution set to a linear inequality in two variables. Build a function that models a relationship between two quantities.

KEY IDEAS

1. The graph of a linear equation in two variables is a collection of ordered pair solutions in a coordinate plane. It is a graph of a straight line. Often tables of values are used to organize the ordered pairs.

   Example:

   Every year Silas buys fudge at the state fair. He buys two types: peanut butter and chocolate. This year he intends to buy $24 worth of fudge. If chocolate costs $4 per pound and peanut butter costs $3 per pound, what are the different combinations of fudge that he can purchase if he only buys whole pounds of fudge?

   Solution:

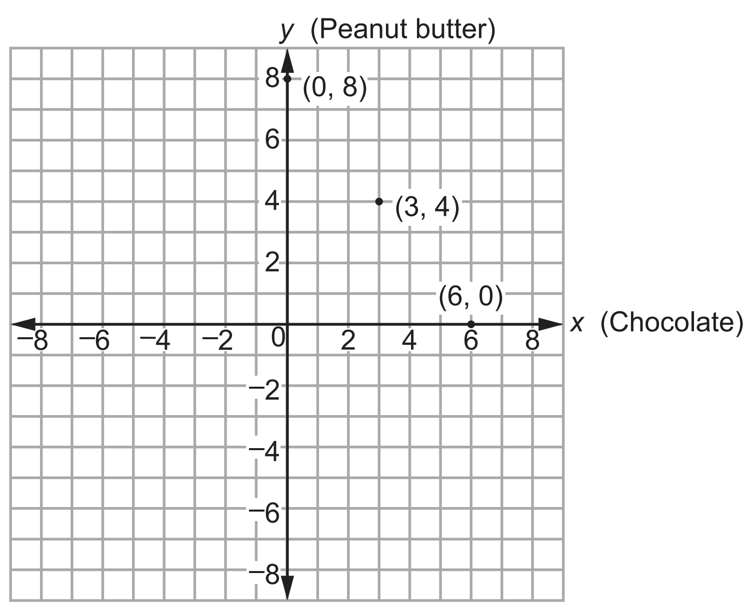
   If we let x be the number of pounds of chocolate and y be the number of pounds of peanut butter, we can use the equation $4x + 3y = 24$. Now we can solve this equation for y to make it easier to complete our table.

$4x + 3y = 24$	Write the original equation.
$4x - 4x + 3y = 24 - 4x$	Addition Property of Equality
$3y = 24 - 4x$	Additive Inverse Property
$\frac{3y}{3} = \frac{24-4x}{3}$	Multiplication Property of Equality
$y = \frac{24-4x}{3}$	Multiplicative Inverse Property

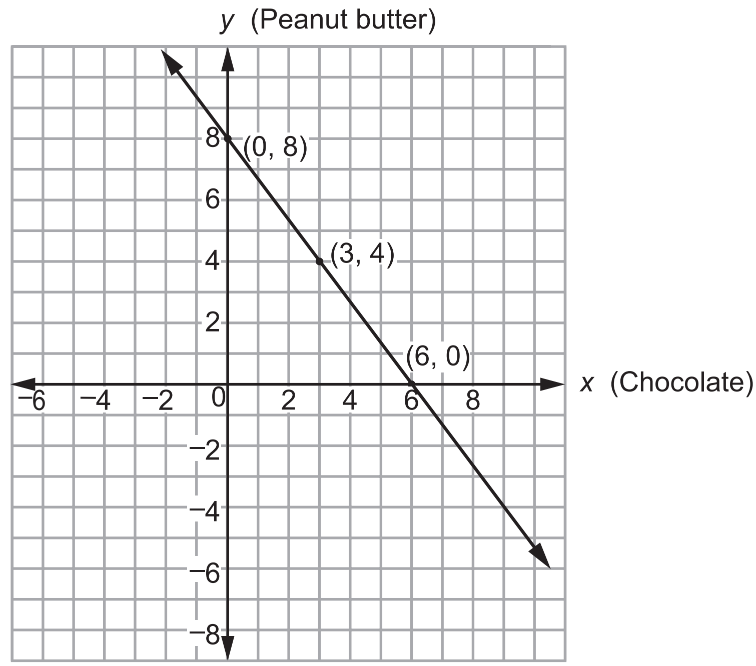
We will only use whole numbers in the table because Silas will only buy whole pounds of fudge.

Chocolate (x)	Peanut butter (y)
0	8
1	~~$6\frac{2}{3}$~~ (not a whole number)
2	~~$5\frac{1}{3}$~~ (not a whole number)
3	4
4	~~$2\frac{2}{3}$~~ (not a whole number)
5	~~$1\frac{1}{3}$~~ (not a whole number)
6	0

The ordered pairs from the table that we want to use are (0, 8), (3, 4), and (6, 0). The graph would look like the one shown below:



Based on the number of points in the graph, there are three possible ways that Silas can buy pounds of fudge: 8 pounds of peanut butter only, 3 pounds of chocolate and 4 pounds of peanut butter, or 6 pounds of chocolate only. Notice that if the points on the graph were joined, they would form a line. If Silas allowed himself to buy partial pounds of fudge, then there would be many more possible combinations. Each combination would total $24 and be represented by a point on the line that contains (0, 8), (3, 4), and (6, 0).



Example:

2. Graph the inequality $x + 2y < 4$.

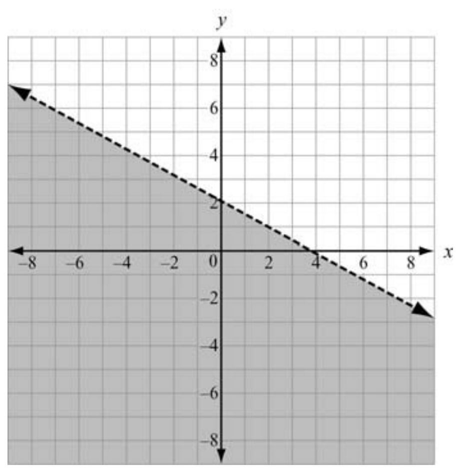
Solution:

The graph looks like a half-plane with a dashed boundary line. The shading is below the line because the points that satisfy the inequality fall below the line. First, graph the line using *x*- and *y*-intercepts. For the *x*-intercept, solve for $y = 0$. For the *y*-intercept, solve for $x = 0$.

$$\begin{aligned} x + 2(0) &< 4 \\ x &< 4 \end{aligned} \qquad \begin{aligned} (0) + 2y &< 4 \\ 2y &< 4 \\ y &< 2 \end{aligned}$$

This gives the points (4, 0) and (0, 2). Since the inequality used the < symbol, use a dotted line through the two points.

Next, decide which side of the boundary line to shade. Use (0, 0) as a test point. Is $0 + 2(0) < 4$? Yes, so (0, 0) is a solution of the inequality. Shade the region below the line. The graph for $x + 2y < 4$ is represented below.



Build a Function That Models a Relationship between Two Quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

> **MGSE9-12.F.BF.1a** Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression "2x + 15" can be described recursively (either in writing or verbally) as "to find out how much money Jimmy will have tomorrow, you add $2 to his total today."* $J_n = J_{n-1} + 2, J_0 = 15$.

MGSE9-12.F.BF.2 Write arithmetic and geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect arithmetic sequences to linear functions and geometric sequences to exponential functions.

KEY IDEAS

1. Modeling a quantitative relationship can be a challenge. But there are some techniques we can use to make modeling easier. Functions can be written to represent the relationship between two variables.

   Example:

   Joe started with $13. He has been saving $2 each week to purchase a baseball glove. The amount of money Joe has depends on how many weeks he has been saving. This means the money in his bank account is the dependent variable and the number of weeks is the independent variable. So, the number of weeks and the amount Joe has saved are related. We can begin with the function $S(x)$, where S is the amount he has saved and x is the number of weeks. Since we know that he started with $13 and that he saves $2 each week we can use a linear model, one where the change is constant.

   A linear model for a function is $f(x) = mx + b$, where m and b are any real numbers and x is the independent variable.

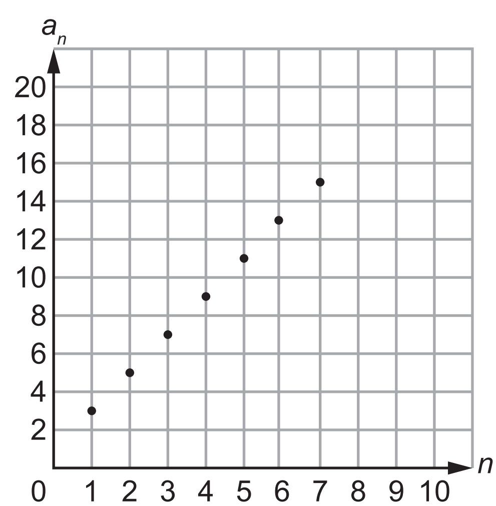
   So, the model is $S(x) = 2x + 13$, which will generate the amount Joe has saved after x weeks.

2. Sometimes the data for a function is presented as a sequence.

Example:

Suppose we know the total number of cookies eaten by Rachel on a day-to-day basis over the course of a week. We might get a sequence like this: 3, 5, 7, 9, 11, 13, 15. There are two ways we could model this sequence. The first would be the explicit way. We would arrange the sequence in a table. Note that *d* in the third row means change, or difference.

n	1	2	3	4	5	6	7
a_n	3	5	7	9	11	13	15
d	—	5 – 3 = 2	7 – 5 = 2	9 – 7 = 2	11 – 9 = 2	13 – 11 = 2	15 – 13 = 2



Since the difference between successive terms of the sequence is constant, namely 2, we can again use a linear model. But this time we do not know the *y*-intercept because there is no zero term ($n = 0$). However, if we work backward, a_0—the term before the first—would be 1, so the starting number would be 1. That leaves us with an explicit formula: $f(n) = 2n + 1$, for $n > 0$ (n is an integer). The explicit formula $a_n = a_1 + d(n - 1)$, where a_1 is the first term and d is the common difference, can be used to find the explicit function. A sequence that can be modeled with a linear function is called an ***arithmetic sequence***.

Another way to look at the sequence is recursively. We need to express term n (a_n) in terms of a previous term (a_{n-1}). Since n is the term, then $n - 1$ is used to represent the previous term. For example, a_3 is the third term, so $a_{3-1} = a_2$ is the second term. Since the constant difference is 2, we know $a_n = a_{n-1} + 2$ for $n > 1$, with $a_1 = 3$.

SAMPLE ITEMS

1. Which function represents the sequence?

n	1	2	3	4	5	. . .
a_n	3	10	17	24	31	. . .

A. $f(n) = n + 3$
B. $f(n) = 7n - 4$
C. $f(n) = 3n + 7$
D. $f(n) = n + 7$

Correct Answer: B

2. Each week, Tim wants to increase the number of sit-ups he does daily by 2 sit-ups. The first week, he does 15 sit-ups each day.

Write an explicit function in the form $f(n) = mn + b$ to represent the number of sit-ups, $f(n)$, Tim does daily in week n.

Solution:

The difference between the number of daily sit-ups each week is always 2, so this is a linear model with $m = 2$. Since there is no zero term, we take the first term, ($n = 1$), and work backwards by subtracting 2 from 15. This gives us $b = 13$. Therefore, the explicit function is $f(n) = 2n + 13$.

A recursive function in the form $f(n + 1) = f(n) + d$, where $f(1) = a$, can be written for the sit-up problem. What recursive function represents the number of sit-ups, $f(n)$, Tim does daily in week n?

Solution:

Since Tim starts out doing 15 sit-ups each day, $f(1) = 15$. The variable d stands for the difference between the number of daily sit-ups Tim does each week, which is 2. The recursive function will be $f(n + 1) = f(n) + 2$, where $f(1) = 15$.

Understand the Concept of a Function and Use Function Notation

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e., each input value maps to exactly one output value. If *f* is a function, *x* is the input (an element of the domain), and *f*(*x*) is the output (an element of the range). Graphically, the graph is $y = f(x)$.

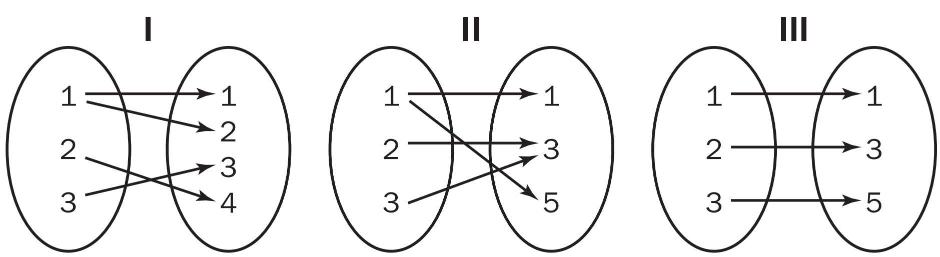
MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. (Generally, the scope of high school math defines this subset as the set of natural numbers 1, 2, 3, 4 . . .) By graphing or calculating terms, students should be able to show how the recursive sequence $a_1 = 7$, $a_n = a_{n-1} + 2$; the sequence $s_n = 2(n - 1) + 7$; and the function $f(x) = 2x + 5$ (when *x* is a natural number) all define the same sequence.

KEY IDEAS

1. There are many ways to show how pairs of quantities are related. Some of them are shown below.

   - ***Mapping Diagrams***



   - ***Sets of Ordered Pairs***

   Set I: {(1, 1), (1, 2), (2, 4), (3, 3)}

   Set II: {(1, 1), (1, 5), (2, 3), (3, 3)}

   Set III: {(1, 1), (2, 3), (3, 5)}

   - ***Tables of Values***

I

x	*y*
1	1
1	2
2	4
3	3

II

x	*y*
1	1
1	5
2	3
3	3

III

x	*y*
1	1
2	3
3	5

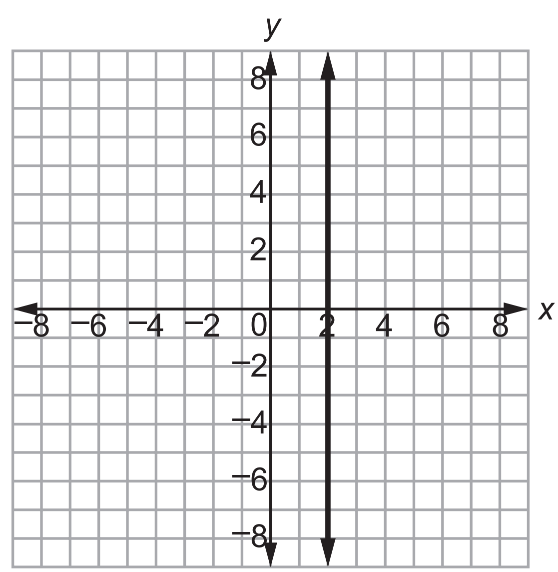
The relationship shown in Mapping Diagram I, Set I, and Table I all represent the same paired numbers. Likewise, Mapping II, Set II, and Table II all represent the same quantities. The same goes for the third group of displays.

Notice the arrows in the mapping diagrams are all arranged from left to right. The numbers on the left side of the mapping diagrams are the same as the *x*-coordinates in the ordered pairs as well as the values in the first column of the tables. Those numbers are called the input values of a quantitative relationship and are known as the ***domain***. The numbers on the right of the mapping diagrams, the *y*-coordinates in the ordered pairs, and the values in the second column of the table are the output, or ***range***. Every number in the domain is assigned to at least one number of the range.

Mapping diagrams, ordered pairs, and tables of values are good to use when there are a limited number of input and output values. There are some instances when the domain has an infinite number of elements to be assigned. In those cases, it is better to use either an algebraic rule or a graph to show how pairs of values are related. Often we use equations as the algebraic rules for the relationships. The domain can be represented by the independent variable and the range can be represented by the dependent variable.

2. A ***function*** is a quantitative relationship where each member of the domain is assigned to exactly one member of the range. Of the relationships on the previous page, only III is a function. In I and II, there were members of the domain that were assigned to two elements of the range. In particular, in I, the value 1 of the domain was paired with 1 and 2 of the range. The relationship is a function if two values in the domain are related to the same value in the range.

   Consider the vertical line $x = 2$. Every point on the line has the same *x*-value and a different *y*-value. So the value of the domain is paired with infinitely many values of the range. This line is not a function. In fact, all vertical lines are not functions.



3. A function can be described using a ***function rule*** that represents an output value, or element of the range, in terms of an input value, or element of the domain.

A function rule can be written in ***function notation*.** Here is an example of a function rule and its notation.

$y = 3x + 5$	y is the output and x is the input.
$f(x) = 3x + 5$	Read as "f of x."
$f(2) = 3(2) + 5$	"f of 2," the value of the function at $x = 2$, is the output when 2 is the input.

Be careful—do not confuse the parentheses used in notation with multiplication.

Functions can also represent real-life situations where $f(15) = 45$ can represent 15 books that cost $45. Functions can have restrictions or constraints that only include whole numbers, such as the situation of the number of people in a class and the number of books in the class. There cannot be half a person or half a book.

Note that all functions have a corresponding graph. The points that lie on the graph of a function are formed using input values, or elements of the domain as the x-coordinates, and output values, or elements of the range as the y-coordinates.

Example:

Given $f(x) = 2x – 1$, find $f(7)$.

Solution:

$f(7) = 2(7) – 1 = 14 – 1 = 13$

Example:

If $g(6) = 3 – 5(6)$, what is $g(x)$?

Solution:

$g(x) = 3 – 5x$

Example:

If $f(–2) = –4(–2)$, what is $f(b)$?

Solution:

$f(b) = –4b$

Example:

Graph $f(x) = 2x - 1$.

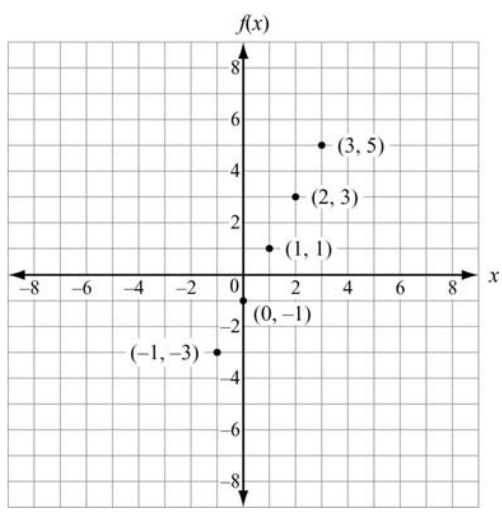
Solution:

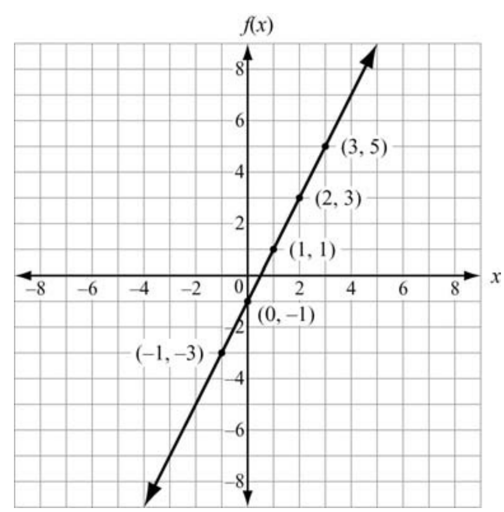
In the function rule $f(x) = 2x - 1$, $f(x)$ is the same as y.

Then we can make a table of x (input) and y (output) values.

x	$f(x)$
–1	–3
0	–1
1	1
2	3
3	5

The values in the rows of the table form ordered pairs. We plot those ordered pairs. If the domain is not specified, we connect the points. If the numbers in the domain are not specified, we assume that they are all real numbers. If the domain is specified such as whole numbers only, then connecting the points is not needed.





4. A ***sequence*** is an ordered list of numbers. Each number in the sequence is called a ***term***. The terms are consecutive or identified as the first term, second term, third term, and so on. The pattern in the sequence is revealed in the relationship between each term and its term number, or in a term's relationship to the previous term in the sequence.

   Example:

   Consider the sequence 3, 6, 9, 12, 15, . . . The first term is 3, the second term is 6, the third term is 9, and so on. The ". . ." at the end of the sequence indicates the pattern continues without end. Can this pattern be considered a function?

   Solution:

   There are different ways of seeing a pattern in the sequence above. The initial term (y-intercept) and the slope can be used to create a table to derive the function. One way is to say each number in the sequence is 3 times the number of its term. For example, the fourth term would be 3 times 4, or 12. Looking at the pattern in this way, all you would need to know is the number of the term, and you could predict the value of the term. The value of each term would be a function of its term number. We could use this relationship to write an algebraic rule for the sequence, $y = 3x$, where x is the number of the term and y is the value of the term. This algebraic rule would only assign one number to each input value from the numbers 1, 2, 3, etc. So, we could write a function for the sequence. We can call the function T and write its rule as $T(n) = 3n$, where n is the term number and 3 is the difference between each term in the sequence, called the common difference. The domain for the function T would be counting numbers. The range would be the value of the terms in the sequence. When an equation with the term number as a variable is used to describe a sequence, we refer to it as the ***explicit formula*** for the sequence, or the ***closed form***. We could also use the common difference and the initial term to find the explicit formula by using $a_n = a_1 + d(n - 1)$, where a_1 is the first term and d is the common difference. We can create the explicit function $T(n) = 3(n - 1) + 3$ for all $n > 0$. The domain for this function would be natural numbers.

   Another way to describe the sequence in the example above is to say each term is three more than the term before it. Instead of using the number of the term, you would need to know a previous term to find a subsequent term's value. We refer to a sequence represented in this form as a ***recursive formula***.

Important Tips

- ✍ Use language carefully when talking about functions. For example, use f to refer to the function as a whole and use $f(x)$ to refer to the output when the input is x.
- ✍ Be sure to check all the terms you are provided with before reaching the conclusion that there is a pattern.

REVIEW EXAMPLES

1. A manufacturer keeps track of her monthly costs by using a “cost function” that assigns a total cost for a given number of manufactured items, x. The function is $C(x) = 5{,}000 + 1.3x$.

   a. What is the reasonable domain of the function?

   b. What is the cost of 2,000 items?

   c. If costs must be kept below $10,000 this month, what is the greatest number of items she can manufacture?

   Solution:

   a. Since x represents a number of manufactured items, it cannot be negative, nor can a fraction of an item be manufactured. Therefore, the domain can only include values that are whole numbers.

   b. Substitute 2,000 for x: $C(2{,}000) = 5{,}000 + 1.3(2{,}000) = \$7{,}600$

   c. Form an inequality:

$$\begin{aligned} C(x) &< 10{,}000 \\ 5{,}000 + 1.3x &< 10{,}000 \\ 1.3x &< 5{,}000 \\ x &< 3{,}846.2, \text{ or } 3{,}846 \text{ items} \end{aligned}$$

2. Consider the first six terms of this sequence: 1, 3, 9, 27, 81, 243, . . .

   a. What is a_1? What is a_3?

   b. What is the reasonable domain of the function?

   c. If the sequence defines a function, what is the range?

   d. What is the common ratio of the function?

   Solution:

   a. a_1 is 1 and a_3 is 9.

   b. The domain is the set of counting numbers: {1, 2, 3, 4, 5, . . .}.

   c. The range is {1, 3, 9, 27, 81, 243, . . .}.

   d. The common ratio is 3.

3. The function $f(n) = -(1 - 4n)$ represents a sequence. Create a table showing the first five terms in the sequence. Identify the domain and range of the function.

Solution:

n	1	2	3	4	5
$f(n)$	3	7	11	15	19

Since the function is a sequence, the domain would be n, the number of each term in the sequence. The set of numbers in the domain can be written as {1, 2, 3, 4, 5, . . .}. Notice that the domain is an infinite set of numbers, even though the table only displays the first five elements.

The range is $f(n)$ or (a_n), the output numbers that result from applying the rule $-1(1 - 4n)$. The set of numbers in the range, which is the sequence itself, can be written as {3, 7, 11, 15, 19, . . .}. This is also an infinite set of numbers, even though the table only displays the first five elements.

SAMPLE ITEMS

1. Look at the sequence in this table.

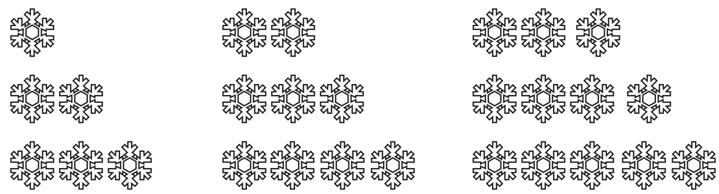
n	1	2	3	4	5	. . .
a_n	–1	1	3	5	7	. . .

Which function represents the sequence?

A. $a_n = a_{n-1} + 1$
B. $a_n = a_{n-1} + 2$
C. $a_n = 2a_{n-1} - 1$
D. $a_n = 2a_{n-1} - 3$

Correct Answer: B

2. Consider this pattern.



Which function represents the sequence that represents the pattern?

A. $a_n = a_{n-1} - 3$
B. $a_n = a_{n-1} + 3$
C. $a_n = 3a_{n-1} - 3$
D. $a_n = 3a_{n-1} + 3$

Correct Answer: B

3. Which function is modeled in this table?

x	$f(x)$
1	8
2	11
3	14
4	17

A. $f(x) = x + 7$
B. $f(x) = x + 9$
C. $f(x) = 2x + 5$
D. $f(x) = 3x + 5$

Correct Answer: D

4. Which explicit formula describes the pattern in this table?

d	*C*
2	6.28
3	9.42
5	15.70
10	31.40

A. $d = 3.14 \times C$
B. $3.14 \times C = d$
C. $31.4 \times 10 = C$
D. $C = 3.14 \times d$

Correct Answer: D

5. If $f(12) = 4(12) - 20$, which function gives $f(x)$?

A. $f(x) = 4x$
B. $f(x) = 12x$
C. $f(x) = 4x - 20$
D. $f(x) = 12x - 20$

Correct Answer: C

Interpret Functions That Arise in Applications in Terms of the Context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior.

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

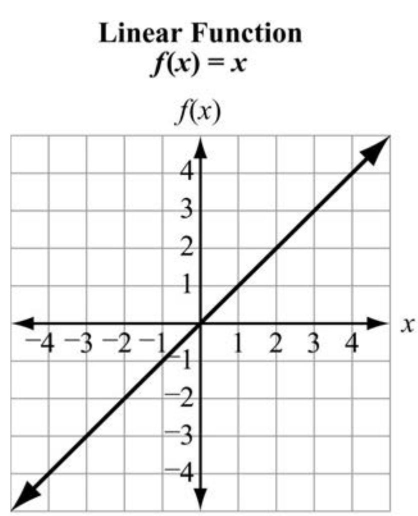
MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

KEY IDEAS

1. By examining the graph of a function, many of its features are discovered. Features include domain and range; *x*- and *y*-intercepts; intervals where the function values are increasing, decreasing, positive, or negative; and rates of change.

   Example:

   Consider the graph of $f(x) = x$. It appears to be an unbroken line and slanted upward.



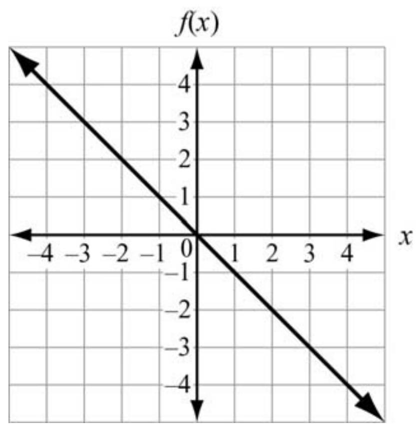
   Some of its key features are

   - Domain: All real numbers
   - Range: All real numbers
   - *x*-intercept: The line appears to intersect the *x*-axis at 0.
   - *y*-intercept: The line appears to intersect the *y*-axis at 0.

- Increasing: for x $(-\infty, \infty)$: as x increases, $f(x)$ increases
- Decreasing: Never
- Positive: $f(x) > 0$ when $x > 0$
  Negative: $f(x) < 0$ when $x < 0$
- Rate of change: 1
- End behavior: decreases as x goes to $-\infty$ and increases as x goes to ∞

Example:

Consider the graph of $f(x) = -x$. It appears to be an unbroken line and slanted downward.



Some of its key features are

- Domain: All real numbers because there is a point on the graph for every possible x-value
- Range: All real numbers because there is a point on the graph that corresponds to every possible y-value
- x-intercept: It appears to intersect the x-axis at 0.
- y-intercept: It appears to intersect the y-axis at 0.
- Increasing: The function does not increase.
- Decreasing: for x $(-\infty, \infty)$
- Positive: $f(x)$ is positive for $x < 0$
  Negative: $f(x)$ is negative for $x > 0$
- Rate of change: -1
- End behavior: increases as x goes to $-\infty$ and decreases as x goes to ∞

2. Other features of functions can be discovered through examining their tables of values. The intercepts may appear in a table of values. From the differences of $f(x)$-values over various intervals, we can tell if a function grows at a constant rate of change.

Example:

Let $h(x)$ be the number of hours it takes a new factory to produce x engines. The company's accountant determines that the number of hours it takes depends on the time it takes to set up the machinery and the number of engines to be completed. It takes 6.5 hours to set up the machinery to make the engines and about 5.25 hours to completely manufacture one engine. The relationship is modeled with the function $h(x) = 6.5 + 5.25x$. Next, the accountant makes a table of values to check his function against his production records. The accountant starts with 0 engines because of the time it takes to set up the machinery.

The realistic domain for the accountant's function would be whole numbers because you cannot manufacture a negative number of engines.

x (number of engines)	$h(x)$ (hours to produce)
0	6.5
1	11.75
2	17
3	22.25
4	27.5
5	32.75
10	59
100	531.5

From the table we can see the y-intercept. The y-intercept is the y-value when $x = 0$. The very first row of the table indicates the y-intercept is 6.5. Since we do not see the number 0 in the $h(x)$ column, we cannot tell from the table whether there is an x-intercept. The x-intercept is the value when $h(x) = 0$.

$$\begin{aligned} h(x) &= 6.5 + 5.25x \\ 0 &= 6.5 + 5.25x \\ -6.5 &= 5.25x \\ -1.24 &= x \end{aligned}$$

The x-value when $y = 0$ is negative, which is not possible in the context of this example.

The accountant's table also gives us an idea of the rate of change of the function. We should notice that as x-values are increasing by 1, the $h(x)$-values are growing by increments of 5.25. There appears to be a constant rate of change when the input values increase by the same amount. The increase from both 3 engines to 4 engines and 4 engines to 5 engines is 5.25 hours. The average rate of change can be calculated by comparing the values in the first or last rows of the table. The increase in number of engines manufactured is 100 – 0, or 100. The increase in hours to produce the engines is 531.5 – 6.5, or 525. The average rate of change is $\frac{525}{100} = 5.25.$ The units for this average rate of change would be hours/engine, which happens to be the exact amount of time it takes to manufacture 1 engine.

Important Tips

- ✍ One method for exploration of a new function could begin by generating a table of values using a variety of numbers from the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure to choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain.
- ✍ You cannot always find exact values from a graph. Always check your answers using the equation.

REVIEW EXAMPLE

1. A company uses the function $V(x) = 28{,}000 - 1{,}750x$ to represent the amount left to pay on a truck, where $V(x)$ is the amount left to pay on the truck, in dollars, and x is the number of months after its purchase. Use the table of values shown below.

x (months)	$V(x)$ ($)
0	28,000
1	26,250
2	24,500
3	22,750
4	21,000
5	19,250

a. What is the y-intercept of the graph of the function in terms of the amount left to pay on the truck?

b. Does the graph of the function have an x-intercept, and if so what does that represent?

c. Does the function increase or decrease?

Solution:

a. From the table, when $x = 0$, $V(x) = 28{,}000$. So, the y-intercept is 28,000, which means at zero months, the amount left to pay on the truck had not yet decreased.

b. Yes, it does have an x-intercept, although it is not shown in the table. The x-intercept is the value of x when $V(x) = 0$.

$$\begin{aligned} 0 &= 28{,}000 - 1{,}750x \\ -28{,}000 &= -1{,}750x \\ 16 &= x \end{aligned}$$

The x-intercept is 16. This means that the truck is fully paid off after 16 months of payments.

c. For $x > 0$, as x increases, $V(x)$ decreases. Therefore, the function decreases.

SAMPLE ITEM

1. **A wild horse runs at a rate of 8 miles an hour for 6 hours. Let *y* be the distance, in miles, the horse travels for a given amount of time, *x*, in hours. This situation can be modeled by a function.**

   Which of these describes the domain of the function?

   A. $0 \leq x \leq 6$
   B. $0 \leq y \leq 6$
   C. $0 \leq x \leq 48$
   D. $0 \leq y \leq 48$

Correct Answer: A

Analyze Functions Using Different Representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

KEY IDEA

1. When working with functions, it is essential to be able to interpret the specific quantitative relationship regardless of the manner of its presentation. Understanding different representations of functions, such as tables, graphs, equations, and verbal descriptions, makes interpreting relationships between quantities easier. Beginning with lines, we will learn how each representation aids our understanding of a function. Almost all lines are functions, except vertical lines, because they assign multiple elements of their range to just one element in their domain. All linear functions can be written in the form $y = mx + b$, where m and b are real numbers and x is a variable to which the function f assigns a corresponding value, $f(x)$.

   Example:

   Consider the linear functions $f(x) = x + 5$, $g(x) = 2x - 5$, and $h(x) = -2x$.

   First, we will make a table of values for each equation. To begin, we need to decide on the domains. In theory, $f(x)$, $g(x)$, and $h(x)$ can accept any number as input. So, the three of them have all real numbers as their domains. But for a table, we can only include a few elements of their domains. We should choose a sample that includes negative numbers, 0, and positive numbers. Place the elements of the domain in the left column, usually in ascending order. Then apply the function rule to determine the corresponding elements in the range. Place them in the right column.

x	*f*(*x*) = *x* + 5
–3	2
–2	3
–1	4
0	5
1	6
2	7
3	8
4	9

x	*g*(*x*) = 2*x* – 5
–3	–11
–2	–9
–1	–7
0	–5
1	–3
2	–1
3	1
4	3

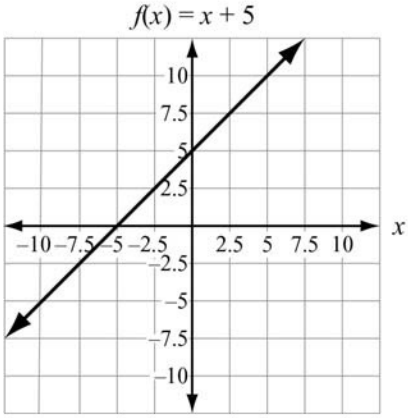
x	*h*(*x*) = –2*x*
–3	6
–2	4
–1	2
0	0
1	–2
2	–4
3	–6
4	–8

We can note several features about the functions just from their tables of values.

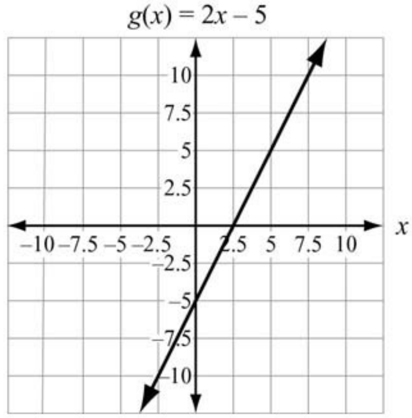
- *f*(*x*) has a *y*-intercept of 5. When *x* is 0, *f*(*x*) = 5. It is represented by (0, 5) on its graph.
- *g*(*x*) has a *y*-intercept of –5. When *x* is 0, *g*(*x*) = –5. It is represented by (0, –5) on its graph.
- *h*(*x*) has a *y*-intercept of 0. When *x* is 0, *h*(*x*) = 0. It is represented by (0, 0) on its graph.
- *h*(*x*) has an *x*-intercept of 0. When *h*(*x*) = 0, *x* = 0. It is represented by (0, 0) on its graph.
- *f*(*x*) has an average rate of change of 1. $\frac{9-2}{4-(-3)}=1$
- *g*(*x*) has an average rate of change of 2. $\frac{3-(-11)}{4-(-3)}=2$
- *h*(*x*) has an average rate of change of –2. $\frac{(-8)-6}{4-(-3)}=-2$

Now we will take a look at the graphs of *f*(*x*), *g*(*x*), and *h*(*x*).

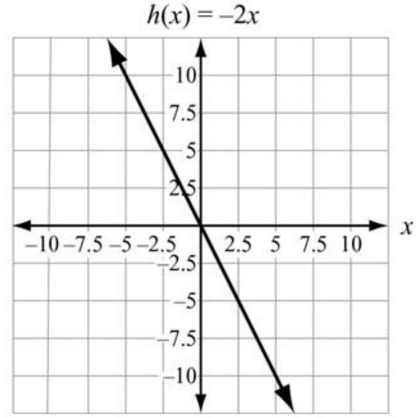
$f(x) = x + 5$



$g(x) = 2x – 5$



$h(x) = –2x$



Their graphs confirm what we already knew about their intercepts and their constant rates of change. To confirm, we can see that $f(x)$ increases by 2.5, as x increases by 2.5, which is a 1 to 1 rate of change. So the slope of $f(x)$ is 1. $g(x)$ increases by 10 as x increases by 5, which is a 2 to 1 rate of change. So the slope is 2. $h(x)$ decreases by 10 as x increases by 2, which is a –2 to 1 rate of change. So the slope is –2. The graphs suggest other information:

- $f(x)$ appears to have positive values for $x > -5$ and negative values for $x < -5$.
- $f(x)$ appears to be always increasing with no maximum or minimum values.
- $g(x)$ appears to have positive values for $x > 2.5$ and negative values for $x < 2.5$.
- $g(x)$ appears to be always increasing with no maximum or minimum values.
- $h(x)$ appears to have positive values for $x < 0$ and negative values for $x > 0$.
- $h(x)$ appears to be always decreasing with no maximum or minimum values.

To confirm these observations, we can work with the equations for the functions. We suspect $f(x)$ is positive for $x > -5$. Since $f(x)$ is positive whenever $f(x) > 0$, write and solve the inequality $x + 5 > 0$ and solve for x. We get $f(x) > 0$ when $x > -5$. We can confirm all our observations about $f(x)$ from working with the equation. Likewise, the observations about $g(x)$ and $h(x)$ can be confirmed using their equations.

Important Tips

✍ Remember that the elements of the domain and the values obtained by substituting them into the function rule form the coordinates of the points that lie on the graph of a function. The domain and range can also be determined by examining the graph of a function by looking for asymptotes on the graph of an exponential function or looking for endpoints or continuity for linear and exponential functions.

✍ Be familiar with important features of a function such as intercepts, domain, range, minimums and maximums, end behavior, asymptotes, and periods of increasing and decreasing values.

REVIEW EXAMPLES

1. What are the key features of the function $p(x) = \frac{1}{2}x - 3$?

   Solution:

   First, notice that the function is linear. The domain for the function is the possible numbers we can substitute for *x*. Since the function is linear and is not related to a real-life situation where certain values are not applicable, the domain is all real numbers. The graphic representation will give us a better idea of its range.

   We can determine the *y*-intercept by finding $p(0)$:

   $$p(0) = \frac{1}{2}(0) - 3 = -3$$

   So, the graph of $p(x)$ will intersect the *y*-axis at (0, –3). To find the *x*-intercept, we have to solve the equation $p(x) = 0$.

   $$\frac{1}{2}x - 3 = 0$$
   $$\frac{1}{2}x = 3$$
   $$x = 6$$

   So, the *x*-intercept is 6. The line intersects the *x*-axis at (6, 0).

   Now we will make a table of values to investigate the rate of change of $p(x)$.

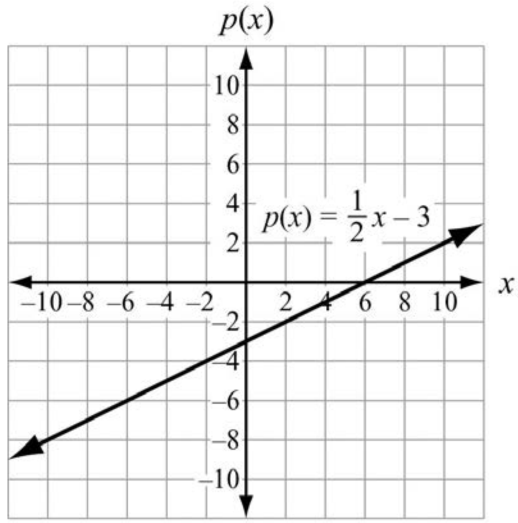
x	$p(x) = \frac{1}{2}x - 3$
–3	$-\frac{9}{2}$
–2	– 4
–1	$-\frac{7}{2}$
0	–3
1	$-\frac{5}{2}$
2	–2
3	$-\frac{3}{2}$
4	–1

Notice the row that contains the values 0 and –3. These numbers correspond to the point where the line intersects the *y*-axis, confirming that the *y*-intercept is –3. Since 0 does not appear in the right column, the coordinates of the *x*-intercept are not in the table of values. We notice that the values in the right column keep increasing by $\frac{1}{2}$. We can calculate the average rate of change.

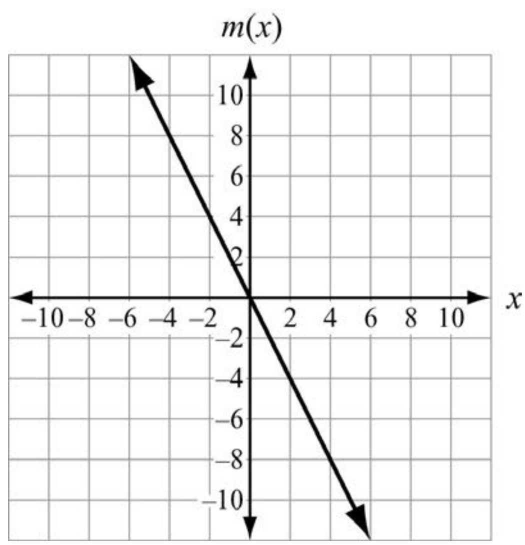
Average rate of change: $\dfrac{-1 - \frac{-9}{2}}{4 - (-3)} = \dfrac{1}{2}$

It turns out that the average rate of change is the same as the incremental differences in the outputs. This confirms that the function $p(x)$ has a constant rate of change. Notice that $\frac{1}{2}$ is the coefficient of x in the function rule.

Now we will examine the graph. The graph shows a line that appears to be always increasing. Since the line has no minimum or maximum value, its range would be all real numbers. The function appears to have positive values for $x > 6$ and negative values for $x < 6$.



2. Compare $p(x) = \frac{1}{2}x - 3$ from the previous example with the function $m(x)$ in the graph below.

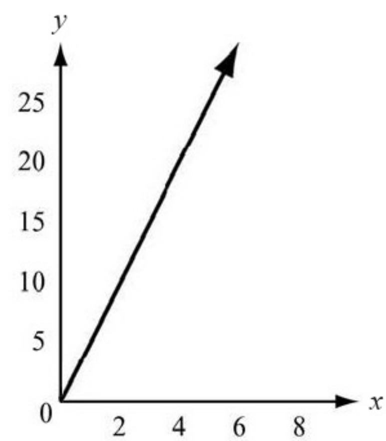


The graph of $m(x)$ intersects both the x- and y-axes at 0. It appears to have a domain of all real numbers and a range of all real numbers. So, $m(x)$ and $p(x)$ have the same domain and range. The graph appears to have a constant rate of change and is decreasing. It has positive values when $x < 0$ and negative values when $x > 0$.

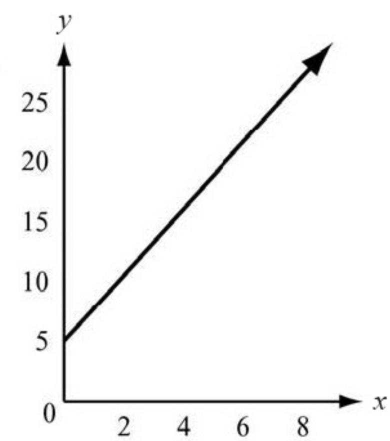
SAMPLE ITEMS

1. To rent a canoe, the cost is \$3 for the oars and life preserver, plus \$5 an hour for the canoe. Which graph models the cost of renting a canoe?

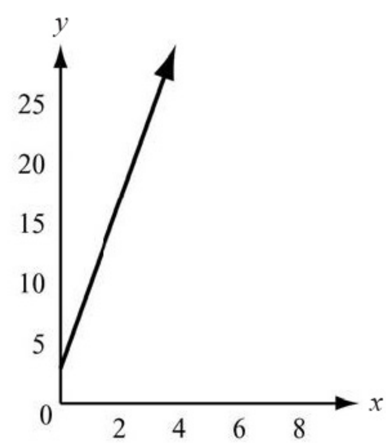
A.



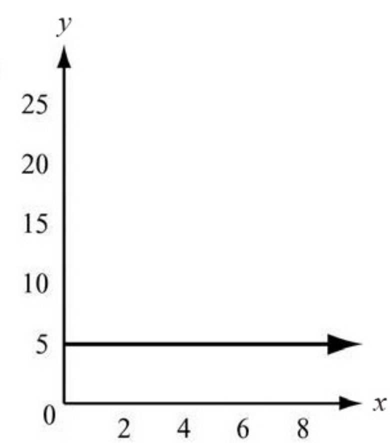
B.



C.

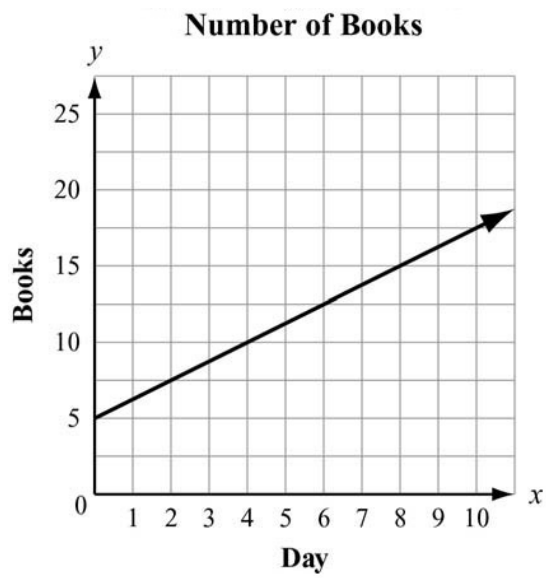


D.



Correct Answer: C

2. Juan and Patti decided to see who could read more books in a month. They began to keep track after Patti had already read 5 books that month. This graph shows the number of books Patti read for the next 10 days and the rate at which she will read for the rest of the month.



If Juan does not read any books before day 4 and he starts reading at the same rate as Patti for the rest of the month, how many books will he have read by day 12?

A. 5
B. 10
C. 15
D. 20

Correct Answer: B

UNIT 3: MODELING AND ANALYZING QUADRATIC FUNCTIONS

This unit investigates quadratic functions. Students study the structure of quadratic expressions and write quadratic expressions in equivalent forms. They solve quadratic equations by inspection, by completing the square, by factoring, and by using the quadratic formula. Students also graph quadratic functions and analyze characteristics of those functions, including end behavior. They write functions for various situations and build functions from other functions.

Interpret the Structure of Expressions

MGSE9-12.A.SSE.2 Use the structure of an expression to rewrite it in different equivalent forms. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

KEY IDEAS

1. An ***algebraic expression*** contains variables, numbers, and operation symbols.
2. A ***term*** in an algebraic expression can be a constant, a variable, or a constant multiplied by a variable or variables. Every term is separated by a plus sign.

   Example:

   The terms in the expression $5x^2 - 3x + 8$ are $5x^2$, $-3x$, and 8.

3. A ***coefficient*** is the constant number that is multiplied by a variable in a term.

   Example:

   The coefficient in the term $7x^2$ is 7.

4. A ***common factor*** is a variable or number that terms can by divided by without a remainder.

   Example:

   The common factors of $30x^2$ and $6x$ are 1, 2, 3, 6, and x.

5. A ***common factor of an expression*** is a number or term that the entire expression can be divided by without a remainder.

   Example:

   The common factor for the expression $3x^2 + 6x - 15$ is 3 because $3x^2 + 6x - 15 = 3(x^2 + 2x - 5)$.

6. If parts of an expression are independent of each other, the expression can be interpreted in different ways.

   Example:

   In the expression $h(3x^2 + 6x - 15)$, the factors h and $(3x^2 + 6x - 15)$ are independent of each other. It can be interpreted as the product of h and a term that does not depend on h.

7. The structure of some expressions can be used to help rewrite them. For example, some fourth-degree expressions are in quadratic form.

   Example:
   $x^2 + 5x + 4 = (x + 4)(x + 1)$

   Example:
   $x^2 - y^2 = (x - y)(x + y)$

REVIEW EXAMPLES

1. Consider the expression $3n^2 + n + 2$.

   a. What is the coefficient of n?

   b. What terms are being added in the expression?

   Solution:

   a. 1

   b. $3n^2$, n, and 2

2. Factor the expression $16a^2 - 81$.

   Solution:
   The expression $16a^2 - 81$ is quadratic in form because it is the difference of two squares ($16a^2 = (4a)^2$ and $81 = 9^2$) and both terms of the binomial are perfect squares. The difference of squares can be factored as:

   $x^2 - y^2 = (x + y)(x - y)$

$16a^2 - 81$	Original expression
$(4a + 9)(4a - 9)$	Factor the binomial (difference of two squares).

3. Factor the expression $12x^2 + 14x - 6$.

   Solution:

$12x^2 + 14x - 6$	Original expression
$2(6x^2 + 7x - 3)$	Factor the trinomial (common factor).
$2(3x - 1)(2x + 3)$	Factor.

SAMPLE ITEMS

1. **Which expression is equivalent to $121x^2 - 64y^2$?**
   - **A.** $(11x - 16y)(11x + 16y)$
   - **B.** $(11x - 16y)(11x - 16y)$
   - **C.** $(11x + 8y)(11x + 8y)$
   - **D.** $(11x + 8y)(11x - 8y)$

Correct Answer: D

2. **What is a common factor for the expression $24x^2 + 16x + 144$?**
   - **A.** 16
   - **B.** $8x$
   - **C.** $3x^2 + 2x + 18$
   - **D.** $8(x - 2)(3x^2 + 9)$

Correct Answer: C

3. **Which of these shows the complete factorization of $6x^2y^2 - 9xy - 42$?**
   - **A.** $3(2xy^2 - 7)(xy^2 + 2)$
   - **B.** $(3xy + 6)(2xy - 7)$
   - **C.** $3(2xy - 7)(xy + 2)$
   - **D.** $(3xy^2 + 6)(2xy^2 - 7)$

Correct Answer: C

Write Expressions in Equivalent Forms to Solve Problems

MGSE9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

MGSE9-12.A.SSE.3a Factor any quadratic expression to reveal the zeros of the function defined by the expression.

MGSE9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum and minimum value of the function defined by the expression.

KEY IDEAS

1. The ***zeros***, ***roots***, or ***x-intercepts*** of a function are the values of the variable that make the function equal to zero. When the function is written in factored form, the Zero Product Property can be used to find the zeros of the function. The Zero Product Property states that if the product of two factors is zero, then one or both of the factors must be zero. So, the zeros of the function are the values that make either factor equal to zero.

   Example:

   $x^2 - 7x + 12 = 0$ Original equation
   $(x - 3)(x - 4) = 0$ Factor.

   Set each factor equal to zero and solve.

   $x - 3 = 0$ $x - 4 = 0$
   $x = 3$ $x = 4$

   The zeros of the function $y = x^2 - 7x + 12$ are $x = 3$ and $x = 4$.

2. To ***complete the square*** of a quadratic function means to write a function as the square of a sum. The standard form for a quadratic expression is $\boldsymbol{ax^2 + bx + c}$, where $a \neq 0$. When $a = 1$, completing the square of the function $x^2 + bx = d$ gives $\left(x + \frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$. To complete the square when the value $a \neq 1$, factor the value of a from the expression.

   Example:

   To complete the square, take half of the coefficient of the x-term, square it, and add it to both sides of the equation.

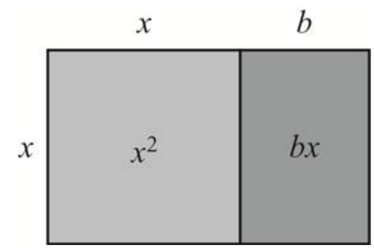
   $x^2 + bx = d$ Original expression

   $x^2 + bx + \left(\frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$ The coefficient of x is b. Half of b is $\frac{b}{2}$. Add the square of $\frac{b}{2}$ to both sides of the equation.

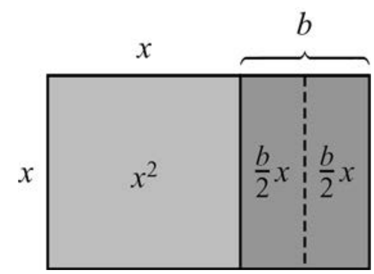
   $\left(x + \frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$ The expression on the left side of the equation is a perfect square trinomial. Factor to write it as a binomial squared.

This figure shows how a model can represent completing the square of the expression $x^2 + bx$, where b is positive.

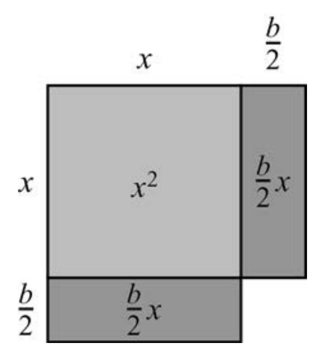
This model represents the expression $x^2 + bx$. To complete the square, create a model that is a square.



Split the rectangle for bx into two rectangles that represent $\frac{b}{2}x$.

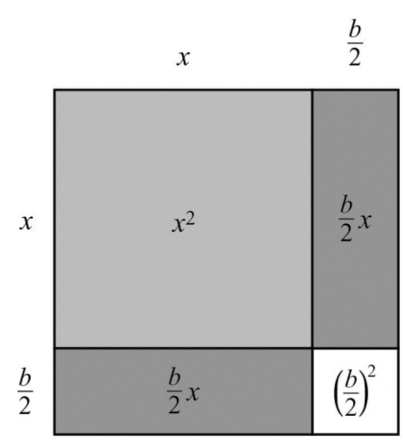


Rearrange the two rectangles that represent $\frac{b}{2}x$.



The missing piece of the square measures $\frac{b}{2}$ by $\frac{b}{2}$. Add and subtract $\left(\frac{b}{2}\right)^2$ to complete the model of the square. The large square has a side length of $x + \frac{b}{2}$, so this model represents $\left(x + \frac{b}{2}\right)^2$.

$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.



Examples:

Complete the square:

$x^2 + 3x + 7$

$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + 7 - \left(\frac{3}{2}\right)^2$

$\left(x + \frac{3}{2}\right)^2 + \frac{19}{4}$

Complete the square:

$x^2 + 3x + 7 = 0$

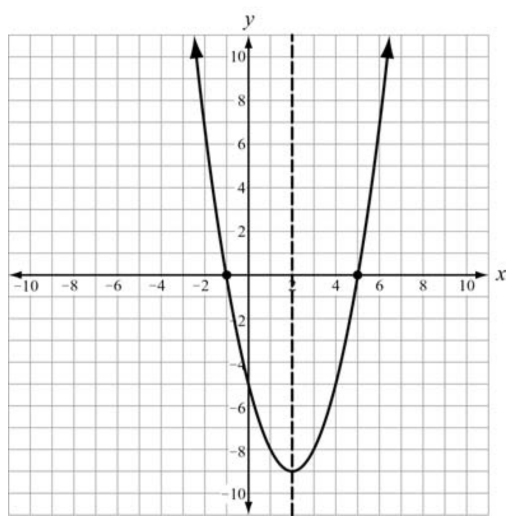
$x^2 + 3x + \left(\frac{3}{2}\right)^2 = -7 + \left(\frac{3}{2}\right)^2$

$\left(x + \frac{3}{2}\right)^2 = -\frac{19}{4}$

3. Every quadratic function has a ***minimum*** or a ***maximum***. This minimum or maximum is located at the ***vertex*** (h, k). The vertex (h, k) also identifies the ***axis of symmetry*** and the minimum or maximum value of the function. The axis of symmetry is $x = h$.

Example:

The quadratic equation $f(x) = x^2 - 4x - 5$ is shown in this graph. The minimum of the function occurs at the vertex $(2, -9)$. The zeros or x-intercepts of the function are $(-1, 0)$ and $(5, 0)$. The axis of symmetry is $x = 2$.



4. The ***vertex form*** of a quadratic function is $f(x) = a(x - h)^2 + k$ where (h, k) is the vertex. One way to convert an equation from standard form to vertex form is to complete the square.
5. The vertex of a quadratic function can also be found by using the ***standard form*** and determining the value $\frac{-b}{2a}$. The vertex is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Important Tips

- When you complete the square, make sure you are only changing the form of the expression and not changing the value.
- When completing the square in an **expression**, add **and subtract** half of the coefficient of the x-term squared.
- When completing the square in an **equation**, add half of the coefficient of the x-term squared **to both sides of the equation.**

REVIEW EXAMPLES

1. Write $f(x) = 2x^2 + 12x + 1$ in vertex form.

Solution Method 1:

The function is in standard form, where $a = 2$, $b = 12$, and $c = 1$.

$2x^2 + 12x + 1$	Original expression
$2(x^2 + 6x) + 1$	Factor out 2 from the quadratic and linear terms.
$2\left(x^2 + 6x + \left(3\right)^2 - \left(3\right)^2\right) + 1$	Add and subtract the square of half of the coefficient of the linear term.
$2\left(x^2 + 6x + \left(3\right)^2\right) - 2(9) + 1$	Remove the subtracted term from the parentheses. Remember to multiply by a.
$2\left(x^2 + 6x + \left(3\right)^2\right) - 17$	Combine the constant terms.
$2\left(x + 3\right)^2 - 17$	Write the perfect square trinomial as a binomial squared.

The vertex of the function is (–3, –17).

Solution Method 2:

The vertex of a quadratic function can also be found by writing the polynomial in standard form and determining the value of $\frac{-b}{2a}$. The vertex is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

For $f(x) = 2x^2 + 12x + 1$, $a = 2$, $b = 12$, and $c = 1$.

$$\frac{-b}{2a} = \frac{-12}{2(2)} = \frac{-12}{4} = -3$$

$$\begin{aligned} f(-3) &= 2(-3)^2 + 12(-3) + 1 \\ &= 2(-3)^2 - 36 + 1 \\ &= 18 - 36 + 1 \end{aligned}$$

The vertex of the function is (–3, –17).

2. The function $h(t) = -t^2 + 8t + 2$ represents the height, in feet, of a stream of water being squirted out of a fountain after t seconds. What is the maximum height of the water?

Solution:

The function is in standard form, where $a = -1$, $b = 8$, and $c = 2$.

The x-coordinate of the vertex is $\frac{-b}{2a} = \frac{-8}{2(-1)} = 4$.

The y-coordinate of the vertex is $h(4) = -(4)^2 + 8(4) + 2 = 18$.

The vertex of the function is (4, 18). So, the maximum height of the water occurs at 4 seconds and is 18 feet.

3. What are the zeros of the function represented by the quadratic expression $x^2 + 6x - 27$?

Solution:

Factor the expression: $x^2 + 6x - 27 = (x + 9)(x - 3)$.

	x + 9	
x – 3	x^2	$9x$
	$-3x$	-27

Set each factor equal to 0 and solve for x.

$x + 9 = 0$ $\qquad$ $x - 3 = 0$

$x = -9$ $\qquad$ $x = 3$

The zeros are $x = -9$ and $x = 3$. This means that $f(-9) = 0$ and $f(3) = 0$.

4. What are the zeros of the function represented by the quadratic expression $2x^2 - 5x - 3$?

Solution:

Factor the expression: $2x^2 - 5x - 3 = (2x + 1)(x - 3)$.

Set each factor equal to 0 and solve for x.

$2x + 1 = 0$ $\qquad$ $x - 3 = 0$

$x = -\frac{1}{2}$ $\qquad$ $x = 3$

The zeros are $x = -\frac{1}{2}$ and $x = 3$.

SAMPLE ITEMS

1. **What are the zeros of the function represented by the quadratic expression $2x^2 + x - 3$?**

   A. $x = -\frac{3}{2}$ and $x = 1$

   B. $x = -\frac{2}{3}$ and $x = 1$

   C. $x = -1$ and $x = \frac{2}{3}$

   D. $x = -1$ and $x = -\frac{3}{2}$

Correct Answer: A

2. **What is the vertex of the graph of $f(x) = x^2 + 10x - 9$?**

   A. (5, 66)
   B. (5, –9)
   C. (–5, –9)
   D. (–5, –34)

Correct Answer: D

3. **Which of these is the result of completing the square for the expression $x^2 + 8x - 30$?**

   A. $(x + 4)^2 - 30$
   B. $(x + 4)^2 - 46$
   C. $(x + 8)^2 - 30$
   D. $(x + 8)^2 - 94$

Correct Answer: B

4. **The expression $-x^2 + 70x - 600$ represents a company's profit for selling x items. For which number(s) of items sold is the company's profit equal to $0?**

   A. 0 items
   B. 35 items
   C. 10 items and 60 items
   D. 20 items and 30 items

Correct Answer: C

Create Equations That Describe Numbers or Relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, and exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P\left(1 + \frac{r}{n}\right)^{nt}$ has multiple variables.)

MGSE9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest using the same reasoning as in solving equations. *Examples: Rearrange Ohm's law $V = IR$ to highlight resistance R; Rearrange area of a circle formula $A = \pi r^2$ to highlight the radius r.*

KEY IDEAS

1. Quadratic equations can be written to model real-world situations.

   Here are some examples of real-world situations that can be modeled by quadratic functions:

   - Finding the area of a shape: Given that the length of a rectangle is 5 units more than the width, the area of the rectangle in square units can be represented by $A = x(x + 5)$, where x is the width and $x + 5$ is the length.
   - Finding the product of consecutive integers: Given a number, n, the next consecutive number is $n + 1$ and the next consecutive even (or odd) number is $n + 2$. The product, P, of two consecutive numbers is $P = n(n + 1)$.
   - Finding the height of a projectile that is dropped: When heights are given in metric units, the equation used is $h(t) = -4.9t^2 + v_o t + h_o$, where v_o is the initial velocity, in meters per second, and h_o is the initial height, in meters. The coefficient -4.9 represents half the force of gravity. When heights are given in customary units, the equation used is $h(t) = -16t^2 + v_o t + h_o$, where v_o is the initial velocity, in feet per second, and h_o is the initial height, in feet. The coefficient -16 represents half the force of gravity. For example, the height, in feet, of a ball thrown with an initial velocity of 60 feet per second and an initial height of 4 feet can be represented by $h(t) = -16t^2 + 60t + 4$, where t is seconds.

2. You can use the properties of equality to solve for a variable in an equation. Use inverse operations on both sides of the equation until you have isolated the variable.

   Example:

   What is the value of r when $S = 0$ for the equation $S = 2\pi r^2 + 2\pi rh$ for r?

Solution:

First, factor the expression $2\pi r^2 + 2\pi rh$.

$$2\pi r(r + h)$$

Next, set each factor equal to 0.

$$2\pi r = 0, \qquad r + h = 0$$
$$r = 0, \qquad r = -h$$

3. To graph a quadratic equation, find the vertex of the graph and the zeros of the equation. The axis of symmetry goes through the vertex and divides the graph into two sides that are mirror images of each other. To draw the graph, you can plot points on one side of the parabola and use symmetry to find the corresponding points on the other side of the parabola.

Example:

Graph the quadratic equation $y = x^2 + 5x + 6$.

Solution:

First, we can find the zeros by solving for x when $y = 0$. This is where the graph crosses the x-axis.

$$0 = x^2 + 5x + 6$$
$$0 = (x + 2)(x + 3)$$
$$x + 2 = 0, x + 3 = 0$$

$x = -2, \quad x = -3$; this gives us the points $(-2, 0)$ and $(-3, 0)$.

Next, we can find the axis of symmetry by finding the vertex. The axis of symmetry is the equation $x = -\dfrac{b}{2a}$. To find the vertex, we first find the axis of symmetry.

$$x = -\frac{5}{2(1)} = -\frac{5}{2}$$

Now we can find the value of the y-coordinate of the vertex.

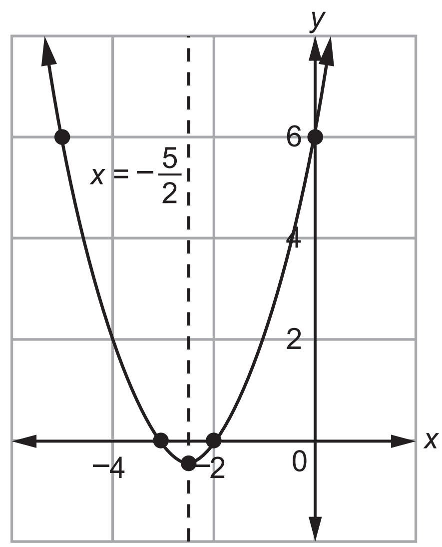
$$\begin{aligned} y &= \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6 \\ &= \frac{25}{4} + \left(-\frac{25}{2}\right) + 6 \\ &= \frac{25}{4} + -\frac{50}{4} + \frac{24}{4} \\ &= -\frac{25}{4} + \frac{24}{4} \\ &= -\frac{1}{4} \end{aligned}$$

So, the vertex is located at $\left(-\dfrac{5}{2}, -\dfrac{1}{4}\right)$.

Next, we can find two more points to continue the curve. We can use the *y*-intercept to find the first of the two points.

$y = (0)^2 + 5(0) + 6 = 6$. The *y*-intercept is at (0, 6).

This point is 2.5 more than the axis of symmetry, so the last point will be 2.5 less than the axis of symmetry. The point 2.5 less than the axis of symmetry with a *y*-value of 6 is (–5, 6).



4. The axis of symmetry is the midpoint for each corresponding pair of *x*-coordinates with the same *y*-value. If (x_1, y) is a point on the graph of a parabola and $x = h$ is the axis of symmetry, then (x_2, y) is also a point on the graph, and x_2 can be found using this equation: $\frac{x_1 + x_2}{2} = h$. In the example above, we can use the zeros (–3, 0) and (–2, 0) to find the axis of symmetry.

$$\frac{-3 + -2}{2} = -\frac{5}{2} = -2.5, \text{ so } x = -2.5$$

REVIEW EXAMPLES

1. The product of two consecutive positive integers is 132.

   a. Write an equation to model the situation.

   b. What are the two consecutive integers?

Solution:

a. Let *n* represent the lesser of the two integers. Then *n* + 1 represents the greater of the two integers. So, the equation is $n(n + 1) = 132$.

b. Solve the equation for *n*.

$n(n + 1) = 132$ Original equation

$n^2 + n = 132$ Distributive Property

$n^2 + n - 132 = 0$ Subtraction Property of Equality

$(n + 12)(n - 11) = 0$ Factor.

Set each factor equal to 0 and solve for n.

$n + 12 = 0$ $\quad$ $n - 11 = 0$

$n = -12$ $\quad$ $n = 11$

Because the two consecutive integers are both positive, $n = -12$ cannot be the solution. So, $n = 11$ is the solution, which means that the two consecutive integers are 11 and 12.

2. The formula for the volume of a cylinder is $V = \pi r^2 h$.

   a. Solve the formula for r.

   b. If the volume of a cylinder is 200π cubic inches and the height of the cylinder is 8 inches, what is the radius of the cylinder?

   Solution:

   a. Solve the formula for r.

   $V = \pi r^2 h$ Original formula

   $\frac{V}{\pi h} = r^2$ Division Property of Equality

   $\pm\sqrt{\frac{V}{\pi h}} = r$ Take the square root of both sides.

   $\sqrt{\frac{V}{\pi h}} = r$ Choose the positive value because the radius cannot be negative.

   b. Substitute 200π for V and 8 for h and evaluate.

   $$r = \sqrt{\frac{V}{\pi h}} = \sqrt{\frac{200\pi}{\pi(8)}} = \sqrt{\frac{200}{8}} = \sqrt{25} = 5$$

   The radius of the cylinder is 5 inches.

3. Graph the function represented by the equation $y = 3x^2 - 6x - 9$.

Solution:

Find the zeros of the equation.

$0 = 3x^2 - 6x - 9$	Set the equation equal to 0.
$0 = 3(x^2 - 2x - 3)$	Factor out 3.
$0 = 3(x - 3)(x + 1)$	Factor.
$0 = (x - 3)(x + 1)$	Division Property of Equality

Set each factor equal to 0 and solve for x.

$x - 3 = 0$	$x + 1 = 0$
$x = 3$	$x = -1$

The zeros are at $x = -1$ and $x = 3$.

Find the vertex of the graph.

$$\frac{-b}{2a} = \frac{-(-6)}{2(3)} = \frac{6}{6} = 1$$

Substitute 1 for x in the original equation to find the y-value of the vertex:

$$\begin{aligned} 3(1)^2 - 6(1) - 9 &= 3 - 6 - 9 \\ &= -12 \end{aligned}$$

Graph the two x-intercepts (3, 0) and (–1, 0) and the vertex (1, –12).

Another descriptive point is the y-intercept. You can find the y-intercept by substituting 0 for x.

$$y = 3x^2 - 6x - 9$$

$$y = 3(0)^2 - 6(0) - 9$$

$$y = -9$$

You can find more points for your graph by substituting x-values into the function. Find the y-value when $x = -2$.

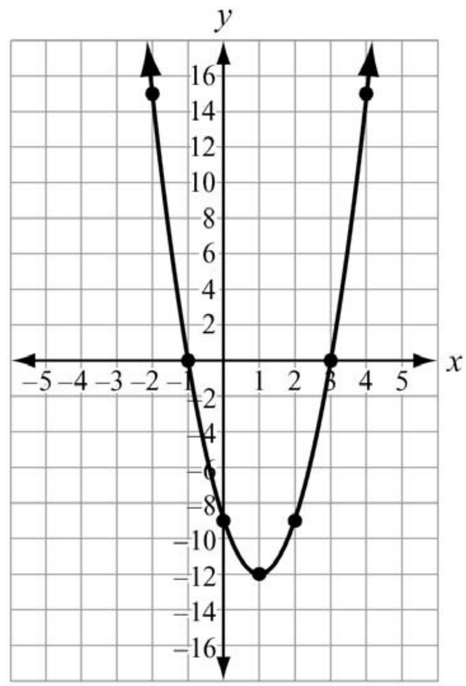
$$y = 3x^2 - 6x - 9$$

$$y = 3(-2)^2 - 6(-2) - 9$$

$$y = 3(4) + 12 - 9$$

$$y = 15$$

Graph the points (0, –9) and (–2, 15). Then use the concept of symmetry to draw the rest of the function. The axis of symmetry is $x = 1$. So, the mirror image of (0, –9) is (2, –9) and the mirror image of (–2, 15) is (4, 15).



SAMPLE ITEMS

1. **A garden measuring 8 feet by 12 feet will have a walkway around it. The walkway has a uniform width, and the area covered by the garden and the walkway is 192 square feet. What is the width of the walkway?**

   A. 2 feet
   B. 3.5 feet
   C. 4 feet
   D. 6 feet

Correct Answer: A

2. **The formula for the area of a circle is $A = \pi r^2$. Which equation shows the formula in terms of r?**

   A. $r = \frac{2A}{\pi}$

   B. $r = \frac{\sqrt{A}}{\pi}$

   C. $r = \sqrt{\frac{A}{\pi}}$

   D. $r = \frac{A}{2\pi}$

Correct Answer: C

Solve Equations and Inequalities in One Variable

MGSE9-12.A.REI.4 Solve quadratic equations in one variable.

MGSE9-12.A.REI.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from $ax^2 + bx + c = 0$.

MGSE9-12.A.REI.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, factoring, completing the square, and the quadratic formula, as appropriate to the initial form of the equation (limit to real number solutions).

KEY IDEAS

1. When quadratic equations do not have a linear term, you can solve the equation by taking the ***square root*** of each side of the equation. Remember, every square root has a positive value and a negative value. Earlier in the guide, we eliminated the negative answers when they represented length or distance.

   Example:

   $3x^2 - 147 = 0$

$3x^2 = 147$	Addition Property of Equality
$x^2 = 49$	Multiplicative Inverse Property
$x = \pm 7$	Take the square root of both sides.

   Check your answers:

   $$\begin{aligned} 3(7)^2 - 147 &= 3(49) - 147 \\ &= 147 - 147 \\ &= 0 \end{aligned}$$

   $$\begin{aligned} 3(-7)^2 - 147 &= 3(49) - 147 \\ &= 147 - 147 \\ &= 0 \end{aligned}$$

2. You can ***factor*** some quadratic equations to find the solutions. To do this, rewrite the equation in standard form set equal to zero ($ax^2 + bx + c = 0$). Factor the expression, set each factor to 0 (by the Zero Product Property), and then solve for x in each resulting equation. This will provide two rational values for x.

   Example:

   $x^2 - x = 12$

$x^2 - x - 12 = 0$	Addition Property of Equality
$(x - 4)(x + 3) = 0$	Factor.

   Set each factor equal to 0 and solve.

   $$\begin{aligned} x - 4 &= 0 & x + 3 &= 0 \\ x &= 4 & x &= -3 \end{aligned}$$

Check your answers:

$$4^2 - 4 = 16 - 4 = 12 \qquad (-3)^2 - (-3) = 9 + 3 = 12$$

3. You can complete the square to solve a quadratic equation. First, write the expression that represents the function in standard form, $ax^2 + bx + c = 0$. Subtract the constant from both sides of the equation: $ax^2 + bx = -c$. Divide both sides of the equation by a: $x^2 + \frac{b}{a}x = \frac{-c}{a}$. Add the square of half the coefficient of the x-term to both sides: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$. Write the perfect square trinomial as a binomial squared: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$. Take the square root of both sides of the equation and solve for x.

Example:

$5x^2 - 6x - 8 = 0$	
$5x^2 - 6x = 8$	Addition Property of Equality
$x^2 - \frac{6}{5}x = \frac{8}{5}$	Division Property of Equality
$x^2 - \frac{6}{5}x + \left(\frac{-3}{5}\right)^2 = \frac{8}{5} + \left(\frac{3}{5}\right)^2$	Addition Property of Equality
$x^2 - \frac{6}{5}x + \left(\frac{-3}{5}\right)^2 = \frac{8}{5} + \frac{9}{25}$	
$\left(x - \frac{3}{5}\right)^2 = \frac{40}{25} + \frac{9}{25}$	Distribution Property
$\left(x - \frac{3}{5}\right)^2 = \frac{49}{25}$	
$x - \frac{3}{5} = \pm\frac{7}{5}$	Take the square root of both sides.
$x = \frac{3}{5} \pm \frac{7}{5}$	Addition Property of Equality
$x = \frac{3}{5} + \frac{7}{5} = \frac{10}{5} = 2;\ x = \frac{3}{5} - \frac{7}{5} = -\frac{4}{5}$	Solve for x for both operations.

4. All quadratic equations can be solved using the quadratic formula. The ***quadratic formula*** is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $ax^2 + bx + c = 0$. The quadratic formula will yield real solutions. We can solve the previous equation using the quadratic formula.

Example:

$5x^2 - 6x - 8 = 0$, where $a = 5$, $b = -6$, and $c = -8$.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(-8)}}{2(5)}$$

$$x = \frac{6 \pm \sqrt{36 - 4(-40)}}{10}$$

$$x = \frac{6 \pm \sqrt{36 - (-160)}}{10}$$

$$x = \frac{6 \pm \sqrt{36 + (160)}}{10}$$

$$x = \frac{6 \pm \sqrt{196}}{10}$$

$$x = \frac{6 \pm 14}{10}$$

$$x = \frac{6 + 14}{10} = \frac{20}{10} = 2;\ \ x = \frac{6 - 14}{10} = \frac{-8}{10} = \frac{-4}{5}$$

Important Tip

✍ While there may be several methods that can be used to solve a quadratic equation, some methods may be easier than others for certain equations.

REVIEW EXAMPLES

1. Solve the equation $x^2 - 10x + 25 = 0$ by factoring.

Solution:
Factor: $x^2 - 10x + 25 = (x - 5)(x - 5)$.

Both factors are the same, so solve the equation:

$x - 5 = 0$

$x = 5$

2. Solve the equation $x^2 - 100 = 0$ by using square roots.

Solution:
Solve the equation using square roots.

$x^2 = 100$	Add 100 to both sides of the equation.
$x = \pm\sqrt{100}$	Take the square root of both sides of the equation.
$x = \pm 10$	Evaluate.

3. Solve the equation $4x^2 - 7x + 3 = 0$ using the quadratic formula.

Solution:
Solve the equation using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $ax^2 - bx + c = 0$.

Given the equation in standard form, the following values will be used in the formula:

$a = 4$, $b = -7$, and $c = 3$

$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(4)(3)}}{2(4)}$	Substitute each value into the quadratic formula.
$x = \dfrac{7 \pm \sqrt{1}}{8}$	Simplify the expression.
$x = \dfrac{7+1}{8} = 1$ and $x = \dfrac{7-1}{8} = \dfrac{6}{8} = \dfrac{3}{4}$	Evaluate.

SAMPLE ITEMS

1. What are the solutions to the equation $2x^2 - 2x - 12 = 0$?

A. $x = -4, x = 3$
B. $x = -3, x = 4$
C. $x = -2, x = 3$
D. $x = -6, x = 2$

Correct Answer: C

2. What are the solutions to the equation $6x^2 - x - 40 = 0$?

A. $x = -\frac{8}{3}, x = -\frac{5}{2}$

B. $x = -\frac{8}{3}, x = \frac{5}{2}$

C. $x = \frac{5}{2}, x = \frac{8}{3}$

D. $x = -\frac{5}{2}, x = \frac{8}{3}$

Correct Answer: D

3. What are the solutions to the equation $x^2 - 5x = 14$?

A. $x = -7$, $x = -2$
B. $x = -14$, $x = -1$
C. $x = -2$, $x = 7$
D. $x = -1$, $x = 14$

Correct Answer: C

4. An object is thrown in the air with an initial velocity of 5 m/s from a height of 9 m. The equation $h(t) = -4.9t^2 + 5t + 9$ models the height of the object in meters after t seconds.

About how many seconds does it take for the object to hit the ground? Round your answer to the nearest tenth of a second.

A. 0.940 second
B. 1.50 seconds
C. 2.00 seconds
D. 9.00 seconds

Correct Answer: C

Build a Function That Models a Relationship Between Two Quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with $15 and earns $2 a day, the explicit expression "2x + 15" can be described recursively (either in writing or verbally) as "to find out how much money Jimmy will have tomorrow, you add $2 to his total today."* $J_n = J_{n-1} + 2$, $J_0 = 15$

KEY IDEAS

1. An ***explicit expression*** contains variables, numbers, and operation symbols and does not use an equal sign to relate the expression to another quantity.
2. A ***recursive process*** can show that a quadratic function has second differences that are equal to one another.

Example:

Consider the function $f(x) = x^2 + 4x - 1$.

This table of values shows five values of the function.

x	$f(x)$
–2	–5
–1	–4
0	–1
1	4
2	11

The first and second differences are shown. The first differences are the differences between the consequence terms. The second differences are the differences between the consequence terms of the first differences.

x	$f(x)$
–2	–5
–1	–4
0	–1
1	4
2	11

First differences

$-4 - (-5) = 1$

$-1 - (-4) = 3$

$4 - (-1) = 5$

$11 - 4 = 7$

Second differences

$3 - 1 = 2$

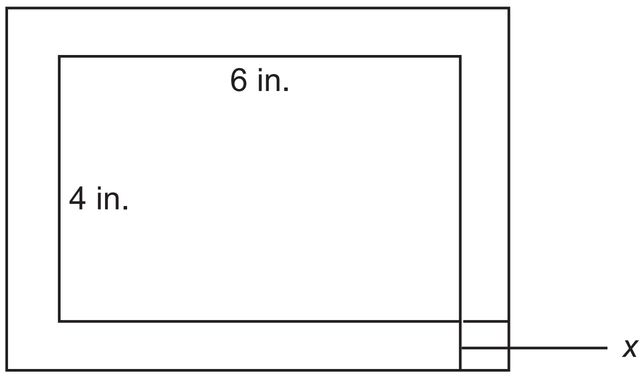
$5 - 3 = 2$

$7 - 5 = 2$

3. A ***recursive function*** is one in which each function value is based on a previous value (or values) of the function.

REVIEW EXAMPLES

1. Annie is framing a photo with a length of 6 inches and a width of 4 inches. The distance from the edge of the photo to the edge of the frame is x inches. The combined area of the photo and frame is 63 square inches.



Note: Image is NOT drawn to scale.

   a. Write a quadratic function to find the distance from the edge of the photo to the edge of the frame.
   b. How wide are the photo and frame together?

   Solution:

   a. The length of the photo and frame is $x + 6 + x = 6 + 2x$. The width of the photo and frame is $x + 4 + x = 4 + 2x$. The area of the frame is $(6 + 2x)(4 + 2x) = 4x^2 + 20x + 24$. Set this expression equal to the area: $63 = 4x^2 + 20x + 24$.
   b. Solve the equation for x.

$$63 = 4x^2 + 20x + 24$$

$$0 = 4x^2 + 20x - 39$$

$$x = -6.5 \text{ or } x = 1.5$$

   Length cannot be negative, so the distance from the edge of the photo to the edge of the frame is 1.5 inches. Therefore, the width of the photo and frame together is $4 + 2x = 4 + 2(1.5) = 7$ inches.

2. A scuba diving company currently charges $100 per dive. On average, there are 30 customers per day. The company performed a study and learned that for every $20 price increase, the average number of customers per day would be reduced by 2.

   a. The total revenue from the dives is the price per dive multiplied by the number of customers. What is the revenue after 4 price increases?
   b. Write a quadratic equation to represent x price increases.
   c. What price would give the greatest revenue?

Solution:

a. Make a table to show the revenue after 4 price increases.

Number of Price Increases	Price per Dive ($)	Number of Customers per Day	Revenue per Day ($)
0	100	30	(100)(30) = 3,000
1	100 + 20(1) = 120	30 – 2(1) = 28	(120)(28) = 3,360
2	100 + 20(2) = 140	30 – 2(2) = 26	(140)(26) = 3,640
3	100 + 20(3) = 160	30 – 2(3) = 24	(160)(24) = 3,840
4	100 + 20(4) = 180	30 – 2(4) = 22	(180)(22) = 3,960

The revenue after 4 price increases is ($180)(22) = $3,960.

b. The table shows a pattern. The price per dive for x price increases is $100 + 20x$. The number of customers for x price increases is $30 - 2x$. So, the equation $y = (100 + 20x)(30 - 2x) = -40x^2 + 400x + 3{,}000$ represents the revenue for x price increases.

c. To find the price that gives the greatest revenue, first find the number of price increases that gives the greatest value. This occurs at the vertex.

Use $\frac{-b}{2a}$ with $a = -40$ and $b = 400$.

$$\frac{-b}{2a} = \frac{-400}{2(-40)} = \frac{-400}{-80} = 5$$

The maximum revenue occurs after 5 price increases.

$100 + 20(5) = 200$

The price of $200 per dive gives the greatest revenue.

3. Consider the sequence 2, 6, 12, 20, 30, ...

   a. What explicit expression can be used to find the next term in the sequence?

   b. What is the tenth term of the sequence?

Solution:

a. The difference between terms is not constant, so the operation involves multiplication. Make a table to try to determine the relationship between the number of the term and the value of the term.

Term number	Term value	Relationship
1	2	$1 \cdot 2$
2	6	$2 \cdot 3$
3	12	$3 \cdot 4$
4	20	$4 \cdot 5$
5	30	$5 \cdot 6$

Notice the pattern: The value of each term is the product of the term number and one more than the term number. So, the expression is $n(n + 1)$ or $n^2 + n$.

b. The tenth term is $n^2 + n = (10)^2 + (10) = 110$.

SAMPLE ITEMS

1. What explicit expression can be used to find the next term in this sequence?

2, 8, 18, 32, 50, . . .

A. $2n$
B. $2n + 6$
C. $2n^2$
D. $2n^2 + 1$

Correct Answer: C

2. The function $s(t) = vt + h - 0.5at^2$ represents the height of an object, *s*, from the ground after the time, *t*, when the object is thrown with an initial velocity of *v* at an initial height of *h* and where *a* is the acceleration due to gravity (32 feet per second squared).

A baseball player hits a baseball 4 feet above the ground with an initial velocity of 80 feet per second. About how long will it take the baseball to hit the ground?

A. 2 seconds
B. 3 seconds
C. 4 seconds
D. 5 seconds

Correct Answer: D

3. A café's annual income depends on *x*, the number of customers. The function $I(x) = 4x^2 - 20x$ describes the café's total annual income. The function $C(x) = 2x^2 + 5$ describes the total amount the café spends in a year. The café's annual profit, $P(x)$, is the difference between the annual income and the amount spent in a year.

Which function describes $P(x)$?

A. $P(x) = 2x^2 - 20x - 5$
B. $P(x) = 4x^3 - 20x^2$
C. $P(x) = 6x^2 - 20x + 5$
D. $P(x) = 8x^4 - 40x^3 - 20x^2 - 100x$

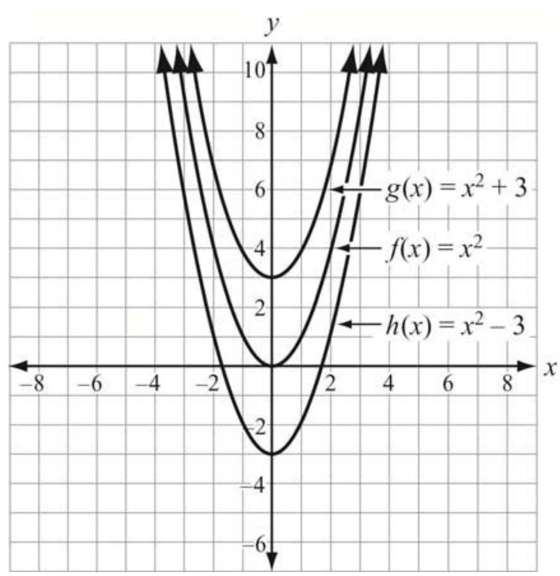
Correct Answer: A

Build New Functions from Existing Functions

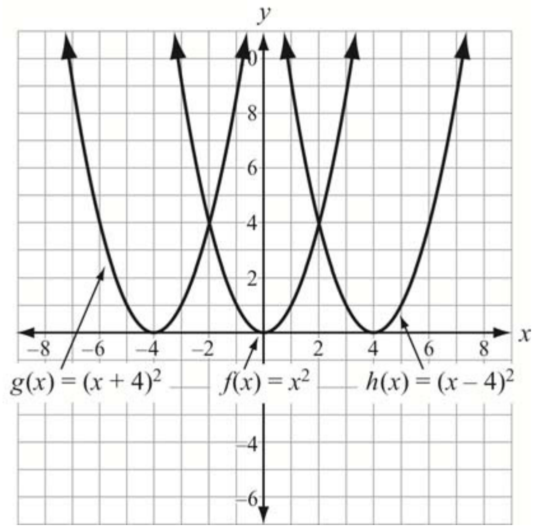
MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

KEY IDEAS

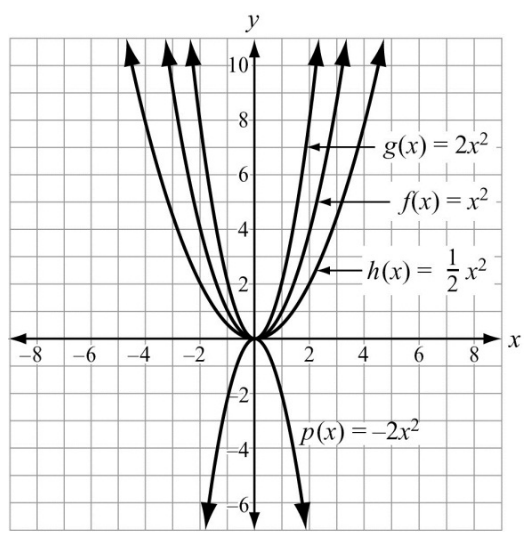
1. A ***parent function*** is the basic function from which all the other functions in a function family are modeled. For the quadratic function family, the parent function is $f(x) = x^2$.
2. For a parent function $f(x)$ and a real number k,
   - the function $f(x) + k$ will move the graph of $f(x)$ up by k units.
   - the function $f(x) – k$ will move the graph of $f(x)$ down by k units.



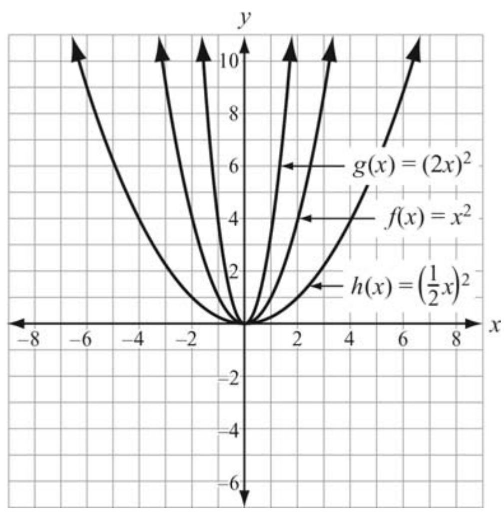
3. For a parent function $f(x)$ and a real number k,
   - the function $f(x + k)$ will move the graph of $f(x)$ left by k units.
   - the function $f(x - k)$ will move the graph of $f(x)$ right by k units.



4. For a parent function $f(x)$ and a real number k,
   - the function $kf(x)$ will vertically stretch the graph of $f(x)$ by a factor of k units for $|k| > 1$.
   - the function $kf(x)$ will vertically shrink the graph of $f(x)$ by a factor of k units for $|k| < 1$.
   - the function $kf(x)$ will reflect the graph of $f(x)$ over the x-axis for negative values of k.

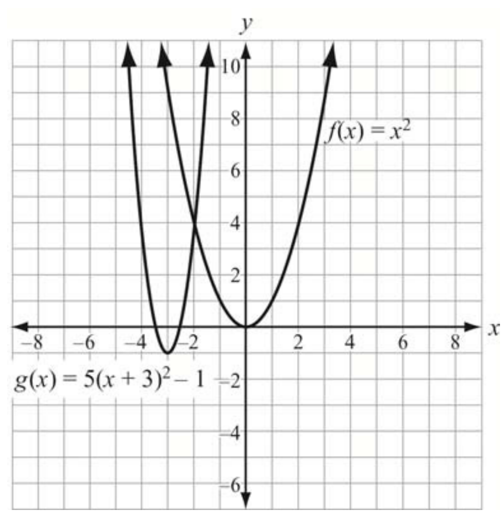


5. For a parent function $f(x)$ and a real number k,
   - the function $f(kx)$ will horizontally shrink the graph of $f(x)$ by a factor of $\frac{1}{k}$ units for $|k| > 1$.
   - the function $f(kx)$ will horizontally stretch the graph of $f(x)$ by a factor of $\frac{1}{k}$ units for $|k| < 1$.
   - the function $f(kx)$ will reflect the graph of $f(x)$ over the y-axis for negative values of k.

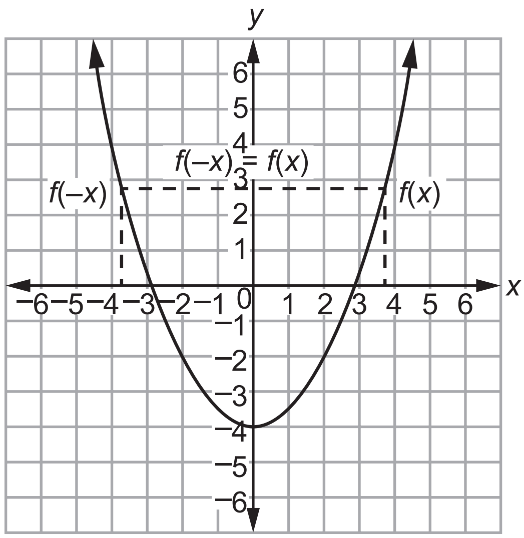


6. You can apply more than one of these changes at a time to a parent function.

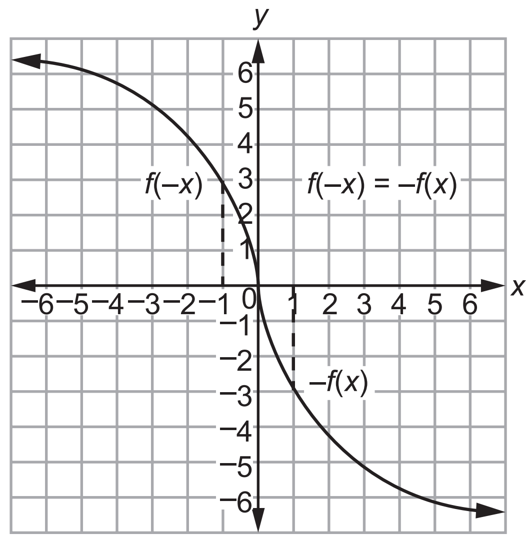
   Example:
   $f(x) = 5(x + 3)^2 - 1$ will translate $f(x) = x^2$ left 3 units and down 1 unit and stretch the function vertically by a factor of 5.



7. Functions can be classified as even or odd.
   - If a graph is symmetric to the y-axis, then it is an ***even function***.
     That is, if $f(-x) = f(x)$, then the function is even.
   - If a graph is symmetric to the origin, then it is an ***odd function***.
     That is, if $f(-x) = -f(x)$, then the function is odd.



Symmetric about the y-axis
Even Function



Symmetric about the origin
Odd Function

Important Tip

✍ Remember that when you change $f(x)$ to $f(x + k)$, move the graph to the **left** when k is positive and to the **right** when k is negative. This may seem different from what you would expect, so be sure to understand why this occurs in order to apply the shift or translation correctly.

REVIEW EXAMPLES

1. Compare the graphs of the following functions to $f(x)$.

   a. $\frac{1}{2}f(x)$

   b. $f(x) - 5$

   c. $f(x - 2) + 1$

   Solution:

   a. The graph of $\frac{1}{2}f(x)$ is a vertical shrink of $f(x)$ by a factor of $\frac{1}{2}$.

   b. The graph of $f(x) - 5$ is a shift or vertical translation of the graph of $f(x)$ down 5 units.

   c. The graph of $f(x - 2) + 1$ is a shift or vertical translation of the graph of $f(x)$ right 2 units and up 1 unit.

2. Is $f(x) = 2x^3 + 6x$ even, odd, or neither? Explain how you know.

Solution:

Substitute $-x$ for x and evaluate:

$$\begin{aligned} f(-x) &= 2(-x)^3 + 6(-x) \\ &= 2(-x)^3 - 6x \\ &= -(2x^3 + 6x) \end{aligned}$$

$f(-x)$ is the opposite of $f(x)$, so the function is odd.

Substitute -3 for x and evaluate:

$$\begin{aligned} f(-3) &= 2(-3)^3 + 6(-3) \\ &= 2(-3)^3 - 18 \\ &= -(2(27) + 18) \\ &= -(72) \end{aligned} \qquad \begin{aligned} f(3) &= 2(3)^3 + 6(3) \\ &= 2(27) + 18 \\ &= 72 \end{aligned}$$

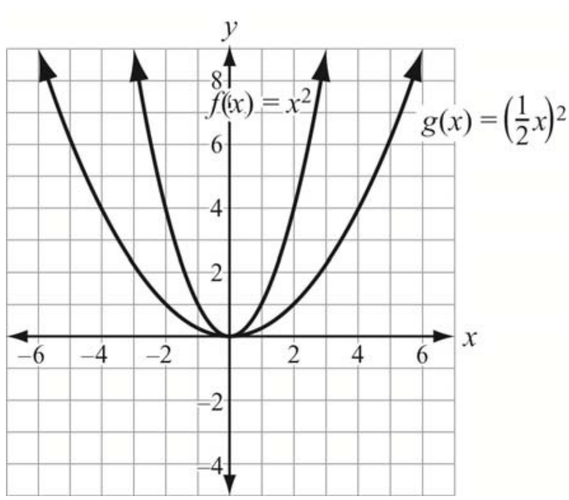
$f(-3)$ is the opposite of $f(3)$, so the function is odd.

3. How does the graph of $f(x)$ compare to the graph of $f\left(\frac{1}{2}x\right)$?

Solution:

The graph of $f\left(\frac{1}{2}x\right)$ is a horizontal stretch of $f(x)$ by a factor of 2. The graphs of $f(x)$ and $g(x) = f\left(\frac{1}{2}x\right)$ are shown.

For example, at $y = 4$, the width of $f(x)$ is 4 and the width of $g(x)$ is 8. So, the graph of $g(x)$ is wider than $f(x)$ by a factor of 2.



SAMPLE ITEMS

1. Which statement BEST describes the graph of $f(x + 6)$?

A. The graph of $f(x)$ is shifted up 6 units.
B. The graph of $f(x)$ is shifted left 6 units.
C. The graph of $f(x)$ is shifted right 6 units.
D. The graph of $f(x)$ is shifted down 6 units.

Correct Answer: B

2. Which of these is an even function?

A. $f(x) = 5x^2 - x$
B. $f(x) = 3x^3 + x$
C. $f(x) = 6x^2 - 8$
D. $f(x) = 4x^3 + 2x^2$

Correct Answer: C

3. Which statement BEST describes how the graph of $g(x) = -3x^2$ compares to the graph of $f(x) = x^2$?

A. The graph of $g(x)$ is a vertical stretch of $f(x)$ by a factor of 3.
B. The graph of $g(x)$ is a reflection of $f(x)$ across the x-axis.
C. The graph of $g(x)$ is a vertical shrink of $f(x)$ by a factor of $\frac{1}{3}$ and a reflection across the x-axis.
D. The graph of $g(x)$ is a vertical stretch of $f(x)$ by a factor of 3 and a reflection across the x-axis.

Correct Answer: D

Interpret Functions That Arise in Applications in Terms of the Context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior.

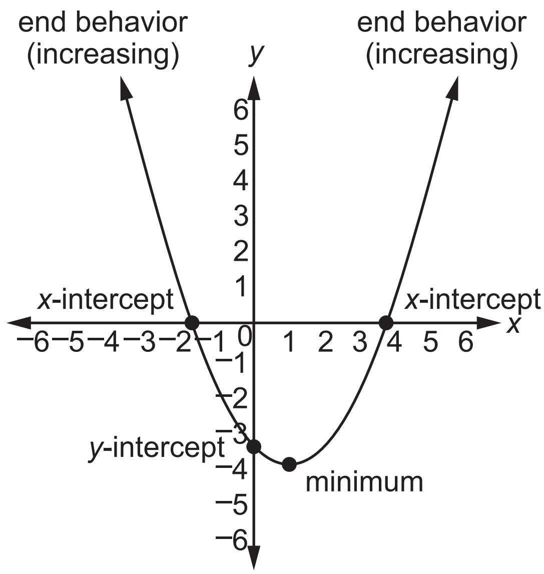
MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

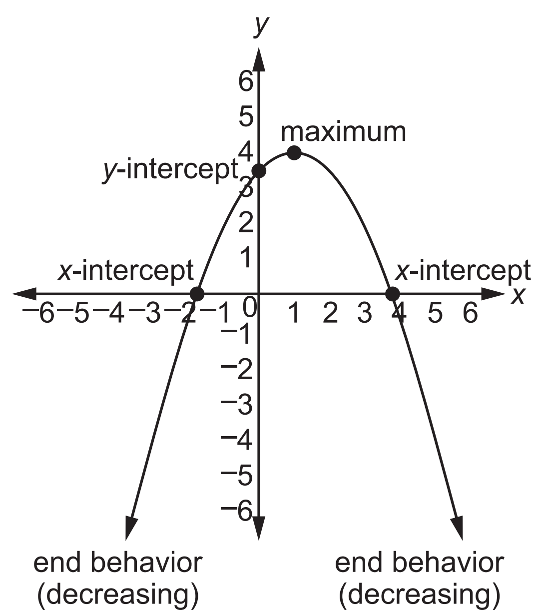
MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

KEY IDEAS

1. An ***x-intercept***, ***root***, or ***zero*** of a function is the *x*-coordinate of a point where the function crosses the *x*-axis. A function may have multiple *x*-intercepts. To find the *x*-intercepts of a quadratic function, set the function equal to 0 and solve for *x*. This can be done by factoring, completing the square, or using the quadratic formula.
2. The ***y-intercept*** of a function is the *y*-coordinate of the point where the function crosses the *y*-axis. A function may have zero *y*-intercepts or one *y*-intercept. To find the *y*-intercept of a quadratic function, find the value of the function when *x* equals 0.
3. A function is ***increasing*** over an interval when the values of *y* increase as the values of *x* increase over that interval. The interval is represented in terms of *x*.
4. A function is ***decreasing*** over an interval when the values of *y* decrease as the values of *x* increase over that interval. The interval is represented in terms of *x*.
5. Every quadratic function has a ***minimum*** or ***maximum***, which is located at the vertex. When the function is written in standard form, the *x*-coordinate of the vertex is $\frac{-b}{2a}$. To find the *y*-coordinate of the vertex, substitute the value of $\frac{-b}{2a}$ into the function and evaluate. When the value of *a* is positive, the graph opens up, and the vertex is the minimum point. When the value of *a* is negative, the graph opens down, and the vertex is the maximum point.
6. The ***end behavior*** of a function describes how the values of the function change as the *x*-values approach negative infinity and positive infinity.
7. The ***domain*** of a function is the set of values for which it is possible to evaluate the function. The domain of a quadratic function is typically all real numbers, although in real-world applications it may only make sense to look at the domain values on a particular interval. For example, time must be a non-negative number.

8. The ***average rate of change*** of a function over a specified interval is the change in the y-value divided by the change in the x-value for two distinct points on a graph. To calculate the average rate of change of a function over the interval from a to b, evaluate the expression $m = \frac{y_2 - y_1}{x_2 - x_1}$.





REVIEW EXAMPLES

1. A ball is thrown into the air from a height of 4 feet at time $t = 0$. The function that models this situation is $h(t) = -16t^2 + 63t + 4$, where t is measured in seconds and h is the height in feet.
   a. What is the height of the ball after 2 seconds?
   b. When will the ball reach a height of 50 feet?
   c. What is the maximum height of the ball?
   d. When will the ball hit the ground?
   e. What domain makes sense for the function?

Solution:

a. To find the height of the ball after 2 seconds, substitute 2 for t in the function.

$h(2) = -16(2)^2 + 63(2) + 4 = -16(4) + 126 + 4 = -64 + 126 + 4 = 66$

The height of the ball after 2 seconds is 66 feet.

b. To find when the ball will reach a height of 50 feet, find the value of t that makes $h(t) = 50$.

$50 = -16t^2 + 63t + 4$

$0 = -16t^2 + 63t - 46$

Use the quadratic formula with $a = -16$, $b = 63$, and $c = -46$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-63 \pm \sqrt{(63)^2 - 4(-16)(-46)}}{2(-16)}$$

$$t = \frac{-63 \pm \sqrt{3969 - 2944}}{-32}$$

$$t = \frac{-63 \pm \sqrt{1025}}{-32}$$

$t \approx 0.97$ or $t \approx 2.97$. So, the ball is at a height of 50 feet after approximately 0.97 second and 2.97 seconds.

c. To find the maximum height, find the vertex of $h(t)$.

The x-coordinate of the vertex is equal to $\frac{-b}{2a}$: $\frac{-63}{2(-16)} \approx 1.97$. To find the y-coordinate, find $h(1.97)$:

$$\begin{aligned} h(1.97) &= -16(1.97)^2 + 63(1.97) + 4 \\ &\approx 66 \end{aligned}$$

The maximum height of the ball is about 66 feet.

d. To find when the ball will hit the ground, find the value of t that makes $h(t) = 0$ (because 0 represents 0 feet from the ground).

$0 = -16t^2 + 63t + 4$

Using the quadratic formula (or by factoring), $t = -0.0625$ or $t = 4$.

Time cannot be negative, so $t = -0.0625$ is not a solution. The ball will hit the ground after 4 seconds.

e. Time must always be non-negative and can be expressed by any fraction or decimal. The ball is thrown at 0 seconds and reaches the ground after 4 seconds. So, the domain $0 \leq t \leq 4$ makes sense for function $h(t)$.

2. This table shows a company's profit, p, in thousands of dollars, over time, t, in months.

Time, t (months)	Profit, p (thousands of dollars)
3	18
7	66
10	123
15	258
24	627

a. Describe the average rate of change in terms of the given context.
b. What is the average rate of change of the profit between 3 and 7 months?
c. What is the average rate of change of the profit between 3 and 24 months?

Solution:

a. The average rate of change represents the rate at which the company earns a profit.

b. Use the expression for average rate of change. Let $x_1 = 3$, $x_2 = 7$, $y_1 = 18$, and $y_2 = 66$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{66 - 18}{7 - 3} = \frac{48}{4} = 12$$

The average rate of change between 3 and 7 months is 12 thousand dollars ($12,000) per month.

c. Use the expression for average rate of change. Let $x_1 = 3$, $x_2 = 24$, $y_1 = 18$, and $y_2 = 627$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{627 - 18}{24 - 3} = \frac{609}{21} = 29$$

The average rate of change between 3 and 24 months is 29 thousand dollars ($29,000) per month.

SAMPLE ITEMS

1. A flying disk is thrown into the air from a height of 25 feet at time $t = 0$. The function that models this situation is $h(t) = -16t^2 + 75t + 25$, where t is measured in seconds and h is the height in feet. What values of t best describe the times when the disk is flying in the air?

A. $0 < t < 5$
B. $0 < t < 25$
C. all real numbers
D. all positive integers

Correct Answer: A

2. Use this table to answer the question.

x	$f(x)$
–2	15
–1	9
0	5
1	3
2	3

What is the average rate of change of $f(x)$ over the interval $-2 \leq f(x) \leq 0$?

A. –10
B. –5
C. 5
D. 10

Correct Answer: B

3. What is the end behavior of the graph of $f(x) = -0.25x^2 - 2x + 1$?

A. As x increases, $f(x)$ increases.
As x decreases, $f(x)$ decreases.

B. As x increases, $f(x)$ decreases.
As x decreases, $f(x)$ decreases.

C. As x increases, $f(x)$ increases.
As x decreases, $f(x)$ increases.

D. As x increases, $f(x)$ decreases.
As x decreases, $f(x)$ increases.

Correct Answer: B

Analyze Functions Using Different Representations

MGSE9-12.F.IF.7 Graph functions expressed algebraically and show key features of the graph both by hand and by using technology.

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

MGSE9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. *For example, compare and contrast quadratic functions in standard, vertex, and intercept forms.*

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.

KEY IDEAS

1. Functions can be represented algebraically, graphically, numerically (in tables), or verbally (by description).

Examples:

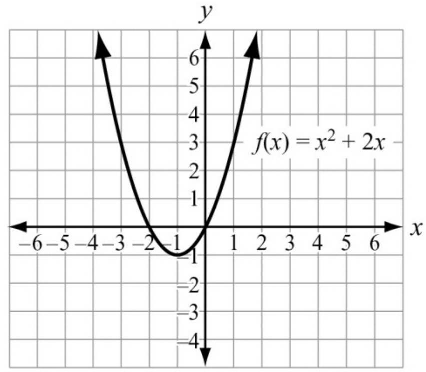
Algebraically: $f(x) = x^2 + 2x$

Verbally (by description): a function that represents the sum of the square of a number and twice the number

Numerically (in a table):

x	$f(x)$
–1	–1
0	0
1	3
2	8

Graphically:



2. You can compare key features of two functions represented in different ways. For example, if you are given an equation of a quadratic function and a graph of another quadratic function, you can calculate the vertex of the first function and compare it to the vertex of the graphed function.

REVIEW EXAMPLES

1. Graph the function $f(x) = x^2 - 5x - 24$.

Solution:

Use the algebraic representation of the function to find the key features of the graph of the function.

Find the zeros of the function.

$0 = x^2 - 5x - 24$ Set the function equal to 0.

$0 = (x - 8)(x + 3)$ Factor.

Set each factor equal to 0 and solve for x.

$$x - 8 = 0 \qquad x + 3 = 0$$
$$x = 8 \qquad x = -3$$

The zeros are at $x = -3$ and $x = 8$.

Find the vertex of the function.

$$x = \frac{-b}{2a} = \frac{-(-5)}{2(1)} = \frac{5}{2} = 2.5$$

Substitute 2.5 for x in the original function to find $f(2.5)$:

$$f(x) = x^2 - 5x - 24$$
$$f(2.5) = (2.5)^2 - 5(2.5) - 24 = 6.25 - 12.5 - 24 = -30.25$$

The vertex is $(2.5, -30.25)$.

Find the y-intercept by finding $f(0)$.

$$f(x) = x^2 - 5x - 24$$
$$f(0) = (0)^2 - 5(0) - 24 = -24$$

The *y*-intercept is (0, –24). Use symmetry to find another point. The line of symmetry is $x = 2.5$.

$$\frac{0 + x}{2} = 2.5$$

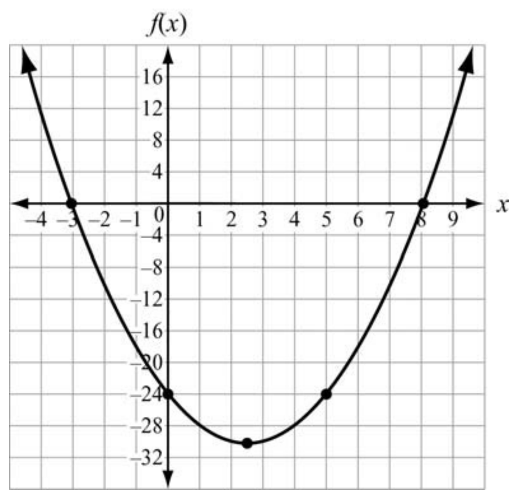
$$0 + x = 5$$

$$x = 5$$

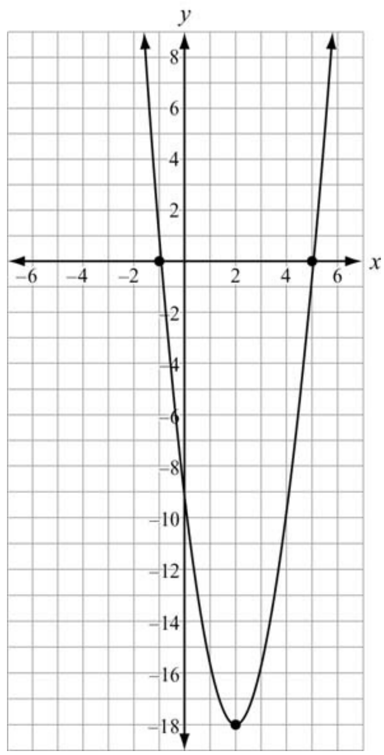
So, point (5, –24) is also on the graph.

Plot the points (–3, 0), (8, 0), (2.5, –30.25), (0, –24), and (5, –24). Draw a smooth curve through the points.

We can also use the value of *a* in the function to determine if the graph opens up or down. In $f(x) = x^2 - 5x - 24$, $a = 1$. Since $a > 0$, the graph opens up.



2. This graph shows a function $f(x)$.



Compare the graph of $f(x)$ to the graph of the function given by the equation $g(x) = 4x^2 + 6x - 18$. Which function has the lesser minimum value? How do you know?

Solution:

The minimum value of a quadratic function is the *y*-value of the vertex.

The vertex of the graph of $f(x)$ appears to be (2, –18). So, the minimum value is –18.

Find the vertex of the function $g(x) = 4x^2 + 6x - 18$.

To find the vertex of $g(x)$, use $\left(\frac{-b}{2a}, g\left(\frac{-b}{2a}\right)\right)$ with $a = 4$ and $b = 6$.

$$x = \frac{-b}{2a} = \frac{-(6)}{2(4)} = \frac{-6}{8} = -0.75$$

Substitute –0.75 for x in the original function $g(x)$ to find $g(-0.75)$:

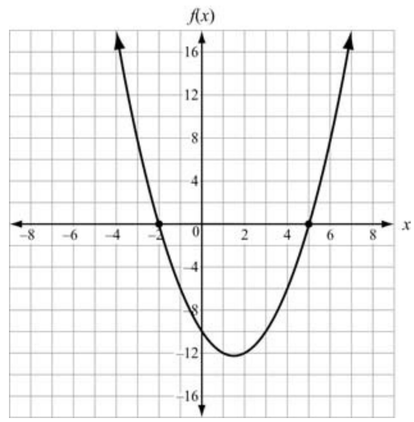
$$\begin{aligned} g(x) &= 4x^2 + 6x - 18 \\ g(-0.75) &= 4(-0.75)^2 + 6(-0.75) - 18 \\ &= 2.25 - 4.5 - 18 \\ &= -20.25 \end{aligned}$$

The minimum value of $g(x)$ is –20.25.

$-20.25 < -18$, so the function $g(x)$ has the lesser minimum value.

SAMPLE ITEMS

1. Use this graph to answer the question.



Which function is shown in the graph?

A. $f(x) = x^2 - 3x - 10$
B. $f(x) = x^2 + 3x - 10$
C. $f(x) = x^2 + x - 12$
D. $f(x) = x^2 - 5x - 8$

Correct Answer: A

2. The function $f(t) = -16t^2 + 64t + 5$ models the height of a ball that was hit into the air, where t is measured in seconds and h is the height in feet.

This table represents the height, $g(t)$, of a second ball that was thrown into the air.

Time, t (in seconds)	Height, $g(t)$ (in feet)
0	4
1	36
2	36
3	4

Which statement BEST compares the length of time each ball is in the air?

A. The ball represented by $f(t)$ is in the air for about 5 seconds, and the ball represented by $g(t)$ is in the air for about 3 seconds.

B. The ball represented by $f(t)$ is in the air for about 3 seconds, and the ball represented by $g(t)$ is in the air for about 5 seconds.

C. The ball represented by $f(t)$ is in the air for about 3 seconds, and the ball represented by $g(t)$ is in the air for about 4 seconds.

D. The ball represented by $f(t)$ is in the air for about 4 seconds, and the ball represented by $g(t)$ is in the air for about 3 seconds.

Correct Answer: D

UNIT 4: MODELING AND ANALYZING EXPONENTIAL FUNCTIONS

In this unit, students focus on exponential equations and functions. Students investigate key features of graphs. They create, solve, and model graphically exponential equations. Students also recognize geometric sequences as exponential functions and write them recursively and explicitly. Given tables, graphs, and verbal descriptions, students interpret key characteristics of exponential functions and analyze these functions using different representations.

Create Equations That Describe Numbers or Relationships

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear, quadratic, and exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase "in two or more variables" refers to formulas like the compound interest formula, in which $A = P\left(1 + \frac{r}{n}\right)^{nt}$ has multiple variables.)

MGSE9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest using the same reasoning as in solving equations. *Examples: Rearrange Ohm's law $V = IR$ to highlight resistance R. Rearrange area of a circle formula $A = \pi r^2$ to highlight the radius r.*

KEY IDEA

1. Exponential equations can be written to model real-world situations. An exponential equation of the form $y = ab^x$ can be applied in many different contexts, including finance, growth, and radioactive decay. For these equations the base, b, must be a positive number and cannot be 1. The coefficient, a, represents the value when $x = 0$, or the initial value.

   Here are some examples of real-world situations that can be modeled by exponential functions:

   - Finding the amount of money in an account with compound interest paid: Use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the principal (the initial amount of money invested that is earning interest), A is the amount of money you would have, with interest, at the end of t years using an annual interest rate of r, and n is the number of compounding periods per year.
   - Finding the population of a city for a given year: Use the formula $P = a \cdot b^x$, where a is the initial population and P is the final population after x years given b rate of growth.

REVIEW EXAMPLES

1. An amount of \$1,000 is deposited into a bank account that pays 4% interest compounded once a year. If there are no other withdrawals or deposits, what will be the balance of the account after 3 years?

   Solution:

   Use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. P is \$1,000, r is 4% or 0.04, n is 1 compounding period a year, and t is 3 years.

   $$A = 1{,}000\left(1 + \frac{0.04}{1}\right)^{1 \cdot 3} = 1{,}000 \times (1.04)^3$$
   $$\approx 1{,}000(1.12486) \approx 1{,}124.86$$

   The balance after 3 years will be \$1,124.86.

2. The city of Arachna has a spider population that has been doubling every year. If there are about 100,000 spiders this year, how many will there be 4 years from now?

   Solution:
   Doubling means multiplying by 2, so use this equation:

   $$S = 100{,}000 \times 2 \times 2 \times 2 \times 2 \text{ or } 100{,}000 \times 2^4$$
   $$S = 1{,}600{,}000$$

   There will be 1,600,000 spiders 4 years from now.

SAMPLE ITEM

1. **A certain population of bacteria has an average growth rate of 2%. The formula for the growth of the bacteria's population is $A = P_0 \cdot 1.02^t$, where P_0 is the original population and t is the time in hours.**

   If you begin with 200 bacteria, about how many bacteria will there be after 100 hours?

   A. 7
   B. 272
   C. 1,478
   D. 20,000

Correct Answer: C

Build a Function That Models a Relationship Between Two Quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with \$15 and earns \$2 a day, the explicit expression "2x + 15" can be described recursively (either in writing or verbally) as "to find out how much money Jimmy will have tomorrow, you add \$2 to his total today."* $J_n = J_{n-1} + 2$, $J_0 = 15$

MGSE9-12.F.BF.2 Write arithmetic and geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect arithmetic sequences to linear functions and geometric sequences to exponential functions.

KEY IDEAS

1. Exponential functions can be used to model quantitative relationships. These functions can be written to represent a relationship between two variables and are sometimes referred to as a geometric sequence.

Example:

Pete withdraws half his savings every week. If he started with \$400, can a rule be written for how much Pete has left each week? We know the amount Pete has left depends on the week. We can start with the amount Pete has, $A(x)$. The amount depends on the week number, x. However, the rate of change is not constant. Therefore, the previous method for finding a function will not work. We could set up the model as

$A(x) = 400 \bullet \frac{1}{2} \bullet \ldots \bullet \frac{1}{2}$ and use $\frac{1}{2}$ as the number of weeks, x.

Or, we can use a power of $\frac{1}{2}$:

$$A(x) = 400 \bullet \left(\frac{1}{2}\right)^x$$

Note that the function assumes Pete had \$400 at week 0 and withdrew half during week 1. The exponential function will generate the amount Pete has after x weeks.

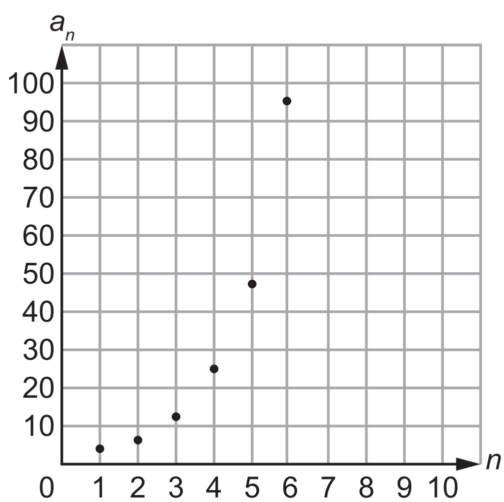
n	1	2	3	4	5	6	7
a_n	200	100	50	25	12.5	6.25	3.125
$\frac{a_n}{a_{n-1}}$	—	$\frac{100}{200} = \frac{1}{2}$	$\frac{50}{100} = \frac{1}{2}$	$\frac{25}{50} = \frac{1}{2}$	$12.5 \div 25 = \frac{1}{2}$	$6.25 \div 12.5 = \frac{1}{2}$	$3.125 \div 6.25 = \frac{1}{2}$

2. Sometimes the data for a function is presented as a sequence that can be modeled exponentially. For a sequence to fit an exponential model, the ratio of successive terms must be constant. In the example below, notice the third row shows a constant ratio between consecutive terms.

Example:

Consider the number of sit-ups Clara does each week as listed in the sequence 3, 6, 12, 24, 48, 96, 192. Clara is doing twice as many sit-ups each successive week. It might be easier to put the sequence in a table to analyze it.

n	1	2	3	4	5	6	7
a_n	3	6	12	24	48	96	192
$\frac{a_n}{a_{n-1}}$	—	$\frac{6}{3}=2$	$\frac{12}{6}=2$	$\frac{24}{12}=2$	$\frac{48}{24}=2$	$\frac{96}{48}=2$	$\frac{192}{96}=2$



It appears as if each term is twice the term before it. But the difference between the terms is not constant. This type of sequence shows exponential growth. The function type is $f(x) = a(b^x)$. In this type of function, b is the base, and b^x is the growth power. For the sequence, the growth power is 2^x because the terms keep doubling. To find b you need to know the first term. The first term is 3. The second term is the first term of the sequence multiplied by the common ratio once. The third term is the first term multiplied by the common ratio twice. Since that pattern continues, our exponential function is $f(x) = 3(2^{x-1})$. The function $f(x) = 3(2^{x-1})$ would be the ***explicit*** or closed form for the sequence. A sequence that can be modeled by an exponential function is a ***geometric sequence***.

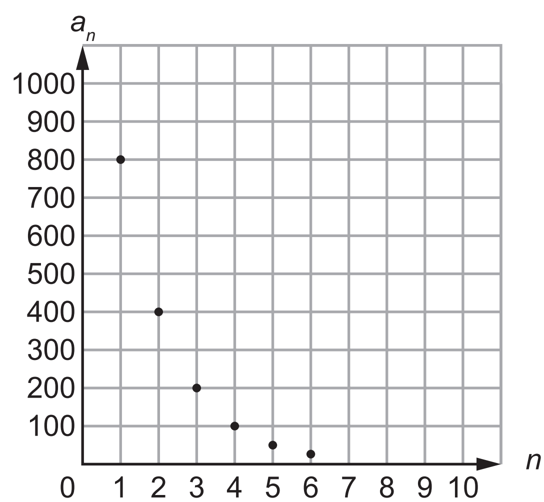
The sequence could also have a recursive rule. Since the next term is twice the previous term, the recursive rule would be $a_n = 2 \cdot a_{n-1}$, with a first term, a_1, of 3.

3. Exponential functions have lots of practical uses. They are used in many real-life situations.

Example:

A scientist collects data on a colony of microbes. She notes these numbers:

Day	1	2	3	4	5	6
a_n	800	400	200	100	50	25
$\frac{a_n}{a_{n-1}}$	—	$\frac{400}{800} = 0.5$	$\frac{200}{400} = 0.5$	$\frac{100}{200} = 0.5$	$\frac{50}{100} = 0.5$	$\frac{25}{50} = 0.5$



Notice a key feature can be seen from the graph. This graph represents exponential decay. Since the ratio between successive terms is a constant 0.5, which is less than 1, the decay factor or common ratio is 0.5. From the table, we can see the initial term, $a_1 = 800$. Since the first term is given, we can use $n - 1$ to represent the subsequent terms. Using the formula $a_n = a_1(r)^{(n-1)}$, we can determine the equation $a_n = 800(0.5)^{(n-1)}$. Since the decay factor is 0.5, the decay rate is 50%. This means the colony of microbes has a half-life of 1 day because it takes 1 day for the number of microbes to decrease by half.

Important Tips

- Examine function values to draw conclusions about the rate of change.
- Keep in mind the general forms of an exponential function.

REVIEW EXAMPLE

1. The temperature of a large tub of water that is currently at 100° decreases by about 10% each hour.

   Part A: Write an explicit function in the form $f(n) = a \cdot b^n$ to represent the temperature, $f(n)$, of the tub of water in n hours.

   Solution:

   The ratio between the changes in temperature each hour is 0.90 since the temperature is decreasing by 0.10. So, this is an exponential model with $0.90 = b$. Since the first term is 100, $a = 100$. Substitute the values into the function $f(n) = a \cdot b^{(n-1)}$, which gives the equation $f(n) = 100(0.90)^{(n-1)}$.

   Part B: A recursive function in the form $f(n) = r(f(n - 1))$, where $f(1) = 100$, can be written for the temperature problem. What recursive function represents the temperature, $f(n)$, of the tub in hour n?

   Solution:

   The temperature starts at $f(1) = 100$. The variable r stands for the ratio between subsequent temperatures, which is 0.90. The recursive function will be $f(n) = (0.90)f(n - 1)$ for all $n > 1$.

SAMPLE ITEMS

1. Which function represents this sequence?

n	1	2	3	4	5	. . .
a_n	6	18	54	162	486	. . .

A. $f(n) = 3^{n-1}$
B. $f(n) = 6^{n-1}$
C. $f(n) = 3(6^{n-1})$
D. $f(n) = 6(3^{n-1})$

Correct Answer: D

2. The points (0, 1), (1, 5), (2, 25), and (3, 125) are on the graph of a function. Which equation represents that function?

A. $f(x) = 2^x$
B. $f(x) = 3^x$
C. $f(x) = 4^x$
D. $f(x) = 5^x$

Correct Answer: D

Build New Functions from Existing Functions

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\ f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

KEY IDEAS

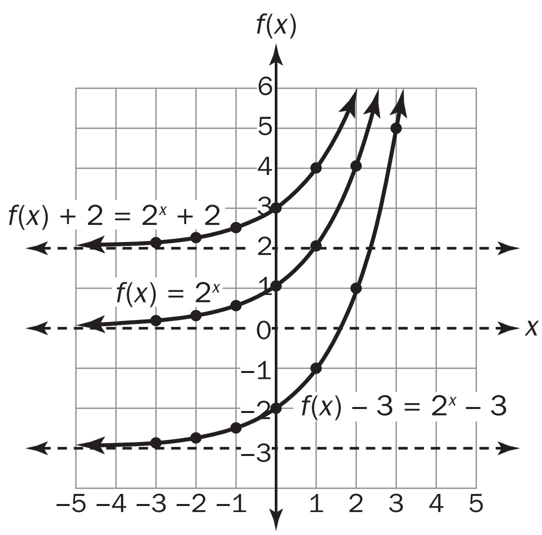
1. Functions can be transformed in many ways. Whenever a function rule is transformed, the transformation affects the graph. One way to change the effect on the characteristics of a graph is to change the constant value of the function. Such adjustments have the effect of shifting the function's graph up or down or right or left. These shifts are called ***translations***. The original curve is moved from one place to another on the coordinate plane the same as translations in geometry.

   Example:

   If $f(x) = 2^x$, how will $g(x) = f(x) + 2$ and $h(x) = f(x) – 3$ compare?

   Solution:

   By using substitution, it makes sense that if $f(x) = 2^x$, then $g(x) = f(x) + b = 2^x + b$. We will compare graphs of $f(x) = 2^x$, $g(x) = f(x) + 2 = 2^x + 2$, and $h(x) = f(x) – 3 = 2^x – 3$.



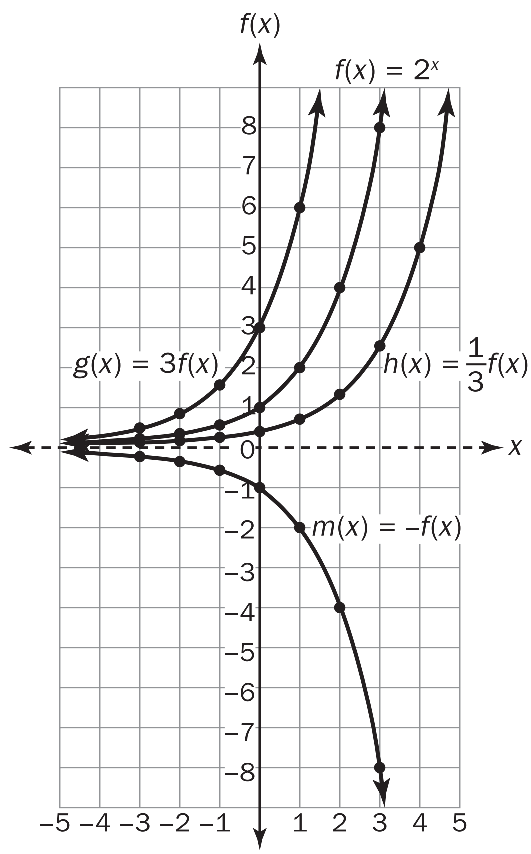
   The curves have not changed shape. Their domains and ranges are unchanged. However, the curves are shifted vertically. The function $g(x)$ is a translation of $f(x) = 2^x$ upward by 2 units. The function $h(x)$ is a translation downward by 3 units. The asymptotes are also shifted vertically.

2. Functions can be adjusted by factors as well as by additions and subtractions. For a function $kf(x)$ or $f(kx)$, k can affect the graph of the function by stretching or shrinking the graph of the function. If the factor is greater than 1, it stretches the graph of the function. If the factor is between 0 and 1, it shrinks the graph of the function. If the factor is –1, it reflects the function over the x-axis.

Example:

If $f(x) = 2^x$, how will $g(x) = 3f(x)$, $h(x) = \frac{1}{3}f(x)$, and $m(x) = -f(x)$ compare?

Solution:



The graphs all have y-intercepts, but the rates of change are affected. They share a horizontal asymptote at $y = 0$.

In summary:

- Adjustments made by adding or subtracting values, either before or after the function, assign values to inputs and cause translations of the graphs.
- Multiplying functions by a constant affects the rate of change of the functions and their graphs.
- Multiplying by a factor of –1 reflects functions over the x-axis.

REVIEW EXAMPLES

1. For the function $f(x) = 3^x$, find the function that represents a 5-unit translation up of the function.

   Solution:

   $f(x) = 3^x + 5$

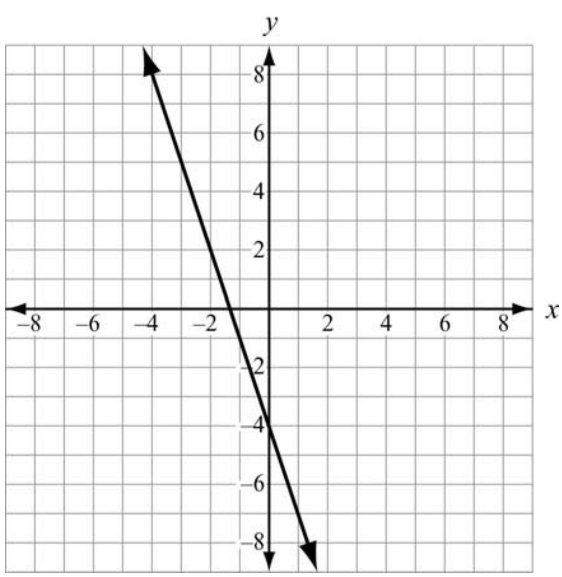
2. Given the function $f(x) = 2^{(x-2)}$, complete each of the following:

   a. Compare $f(x)$ to $3f(x)$.
   b. Compare $f(x)$ to $f(3x)$.
   c. Draw the graph of $-f(x)$.
   d. Which has the fastest growth rate: $f(x)$, $3f(x)$, or $-f(x)$?

   Solution:

   a. $3f(x) = (3)2^{(x-2)}$. So, the curve increases at a higher growth rate and has a different y-intercept.

   b. $f(3x) = 2^{(3x-2)}$. So, the curve increases at a higher rate than $3f(x)$, but has the same y-intercept as $f(x)$.

   c. 

   d. $3f(x)$

SAMPLE ITEMS

1. Which function shows the function $f(x) = 3^x$ being translated 5 units to the left?

A. $f(x) = 3^x - 5$
B. $f(x) = 3^{(x + 5)}$
C. $f(x) = 3^{(x - 5)}$
D. $f(x) = 3^x + 5$

Correct Answer: B

2. Which function shows the function $f(x) = 3^x$ being translated 5 units down?

A. $f(x) = 3^x - 5$
B. $f(x) = 3^{(x + 5)}$
C. $f(x) = 3^{(x - 5)}$
D. $f(x) = 3^x + 5$

Correct Answer: A

Understand the Concept of a Function and Use Function Notation

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of its domain), and $f(x)$ is the output (an element of the range). Graphically, the graph is $y = f(x)$.

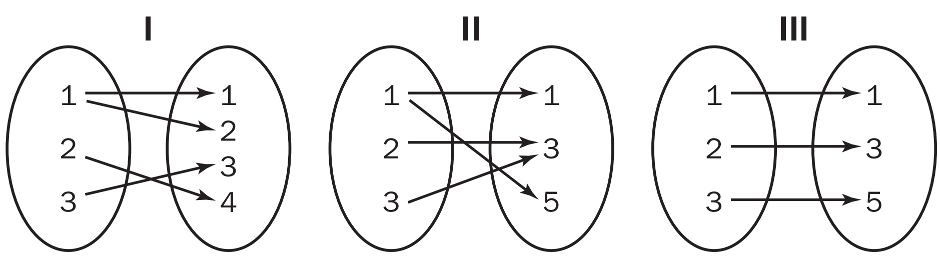
MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. (Generally, the scope of high school math defines this subset as the set of natural numbers 1, 2, 3, 4, ...) By graphing or calculating terms, students should be able to show how the recursive sequence $a_1 = 2$, $a_n = 3(a_{n-1})$; the sequence $a_n = 2(a_{n-1})$; and the function $f(x) = 2(3)^{(x-1)}$ (when x is a natural number) all define the same sequence.

KEY IDEAS

1. There are many ways to show how pairs of quantities are related. Some of them are shown below.

   - ***Mapping Diagrams***



   - ***Sets of Ordered Pairs***

   Set I: {(1, 1), (1, 2), (2, 4), (3, 3)}

   Set II: {(1, 1), (1, 5), (2, 3), (3, 3)}

   Set III: {(1, 1), (2, 3), (3, 5)}

   - ***Tables of Values***

I

x	y
1	1
1	2
2	4
3	3

II

x	y
1	1
1	5
2	3
3	3

III

x	y
1	1
2	3
3	5

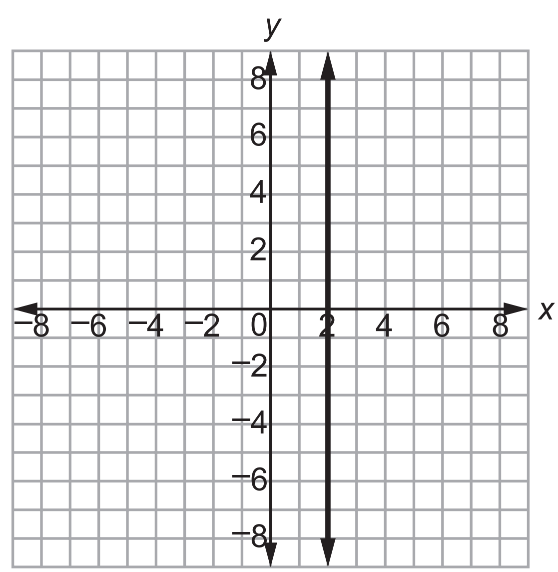
The relationship shown in Mapping Diagram I, Set I, and Table I all represent the same paired numbers. Likewise, Mapping Diagram II, Set II, and Table II all represent the same quantities. The same goes for the third group of displays.

Notice the arrows in the mapping diagrams are all arranged from left to right. The numbers on the left side of the mapping diagrams are the same as the *x*-coordinates in the ordered pairs as well as the values in the first column of the tables. Those numbers are called the input values of a quantitative relationship and are known as the ***domain***. The numbers on the right of the mapping diagrams, the *y*-coordinates in the ordered pairs, and the values in the second column of the table are the output, or ***range***. Every number in the domain is assigned to at least one number of the range.

Mapping diagrams, ordered pairs, and tables of values are good to use when there are a limited number of input and output values. There are some instances when the domain has an infinite number of elements to be assigned. In those cases, it is better to use either an algebraic rule or a graph to show how pairs of values are related. Often we use equations as the algebraic rules for the relationships. The domain can be represented by the independent variable and the range can be represented by the dependent variable.

2. A ***function*** is a quantitative relationship where each member of the domain is assigned to exactly one member of the range. Of the relationships on the previous page, only III is a function. In I and II, there were members of the domain that were assigned to two elements of the range. In particular, in I, the value 1 of the domain was paired with 1 and 2 of the range. The relationship is a function if two values in the domain are related to the same value in the range.

   Consider the vertical line $x = 2$. Every point on the line has the same *x*-value and a different *y*-value. So the value of the domain is paired with infinitely many values of the range. This line is not a function. In fact, all vertical lines are not functions.



3. A function can be described using a ***function rule***, which represents an output value, or element of the range, in terms of an input value, or element of the domain.

   A function rule can be written in ***function notation***. Here is an example of a function rule and its notation.

$y = 2^x$	y is the output and x is the input.
$f(x) = 2^x$	Read as "f of x."
$f(2) = 2^2$	"f of 2," the value of the function at $x = 2$, is the output when 2 is the input.

   Be careful—do not confuse the parentheses used in notation with multiplication.

   Functions can also represent real-life situations, such as where $f(15) = 45$ can represent 15 books that cost $45. Functions can have restrictions or constraints that only include whole numbers, such as the situation of the number of people in a class and the number of books in the class. There cannot be half a person or half a book.

   Note that all functions have a corresponding graph. The points that lie on the graph of a function are formed using input values, or elements of the domain, as the x-coordinates, and output values, or elements of the range, as the y-coordinates.

   Example:

   Given $f(x) = 2(3)^x$, find $f(7)$.

   Solution:

   $f(7) = 2(3)^{(7)} = 2(2{,}187) = 4{,}374$

   Example:

   If $g(6) = 2^{(6)} + 1$, what is $g(x)$?

   Solution:

   $g(x) = 2^x + 1$

   Example:

   If $f(–2) = 4^{(–2)}$, what is $f(b)$?

   Solution:

   $f(b) = 4^{(b)}$

Example:

Graph $f(x) = 4^{(x)} - 5$.

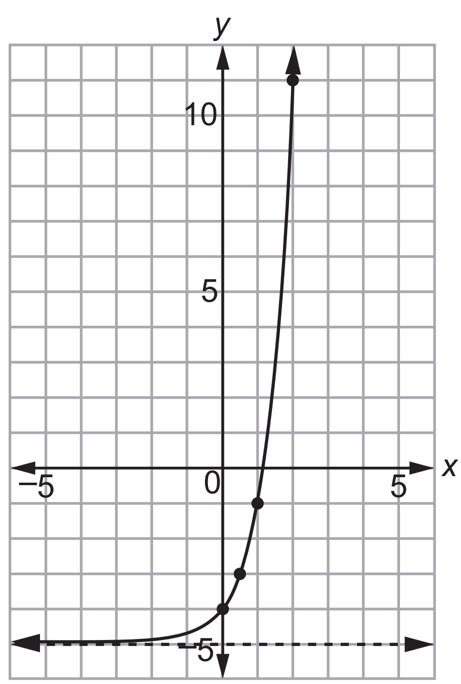
Solution:

In the function rule $f(x) = 4^{(x)} - 5$, $f(x)$ is the same as y.

Then we can make a table of x (input) and y (output) values.

x	$f(x)$
0	–4
0.5	–3
1	–1
2	11

The values in the rows of the table form ordered pairs. We plot those ordered pairs. If the domain is not specified, we connect the points. If the numbers in the domain are not specified, we assume that they are all real numbers. If the domain is specified, such as whole numbers only, then connecting the points is not needed.



4. A ***sequence*** is an ordered list of numbers. Each number in the sequence is called a ***term***. The terms are consecutive or are identified as the first term, second term, third term, and so on. The pattern in the sequence is revealed in the relationship between each term and its term number or in a term's relationship to the previous term in the sequence.

Example:

Consider the sequence: 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$. One way to look at this pattern is to say each successive term is half the term before it, and the first term is 16. With this approach you could easily determine the terms for a limited or ***finite sequence***. Another way would be to notice that each term is 32 times a power of $\frac{1}{2}$. If n represents the number of the term, each term is 32 times $\frac{1}{2}$ raised to the nth power, or $32 \cdot \left(\frac{1}{2}\right)^n$. This approach lends itself to finding an equation with the term number as a variable to describe the sequence. We refer to it as the ***explicit formula*** or the ***closed sequence***. That is, the value of each term would depend on the term number. Using 32 as the initial term, the domain is the set of natural numbers. If 16 is used as the first term, the domain would be whole numbers. The value $\frac{1}{2}$ is multiplied to a term in order to get the following term. This is called the ***common ratio***.

Also, the patterns in sequences can be shown by using tables. For example, this table shows the above sequence:

Term Number (n)	1	2	3	4	5	6	7	8
Value (a_n)	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Notice the numbers in the top row of the table are consecutive counting numbers, starting with one and increasing to the right. The sequence has eight terms, with 16 being the value of the first term and $\frac{1}{8}$ being the value of the eighth term. A sequence with a specific number of terms is finite. If a sequence continues indefinitely, it is called an ***infinite sequence***.

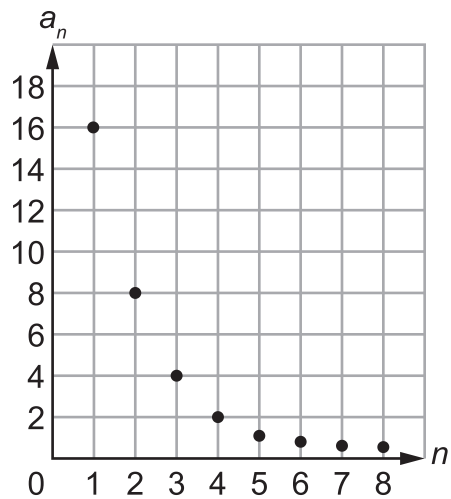
If the nth term of a sequence and the common ratio between consecutive terms is known, you can find the $(n + 1)$th term using the ***recursive formula*** $a_n = a_1 \cdot r^{(n-1)}$, where a_n is the nth term, n is the number of a term, $n - 1$ is the number of the previous term, and r is the common ratio.

Take the sequence 16, 8, 4, 2, 1, . . . as an example. We can find the sixth term of the sequence using the recursive formula.

The common ratio r is $\frac{1}{2}$ and the first term a_1 is 16. So, the sixth term is given by

$$a_6 = a_1 \cdot \left(\frac{1}{2}\right)^{(6-1)} = 16 \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{2}.$$

Also, notice the graph of the sequence. The points are not connected with a curve because the sequence is discrete and not continuous.



Important Tips

- ✍ Use language carefully when talking about functions. For example, use f to refer to the function as a whole and use $f(x)$ to refer to the output when the input is x.
- ✍ The advantages of using an explicit sequence over a recursive sequence is to quickly determine the value of the nth term of the function. However, a recursive sequence helps you see the pattern occurring between sequential terms.

REVIEW EXAMPLES

1. A population of bacteria begins with 2 bacteria on the first day and triples every day. The number of bacteria after x days can be represented by the function $P(x) = 2(3)^x$.

   a. What is the common ratio of the function?
   b. What is a_1 of the function?
   c. Write a recursive formula for the bacteria growth.
   d. What is the bacteria population after 10 days?

   Solution:

   a. The common ratio is 3.
   b. a_1 is 2.
   c. $a_n = a_1(3)^{(n-1)}$
   d. $P(10) = 2(3)^{10} = 2(59{,}049) = 118{,}098$ bacteria after 10 days.

2. Consider the first six terms of the following sequence: 1, 3, 9, 27, 81, 243, . . .

   a. What is a_1? What is a_3?

   b. What is the reasonable domain of the function?

   c. If the sequence defines a function, what is the range?

   d. What is the common ratio of the function?

   Solution:

   a. a_1 is 1 and a_3 is 9.

   b. The domain is the set of counting numbers: {1, 2, 3, 4, 5, . . .}.

   c. The range is {1, 3, 9, 27, 81, 243, . . .}.

   d. The common ratio is 3.

3. The function $f(n) = -(1 - 4^n)$ represents a sequence. Create a table showing the first five terms in the sequence. Identify the domain and range of the function.

   Solution:

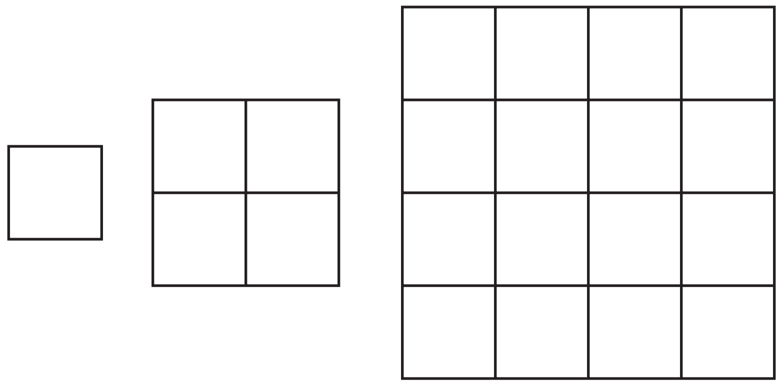
n	1	2	3	4	5
***f*(*n*)**	3	7	63	255	1023

Since the function is a sequence, the domain would be *n*, the number of each term in the sequence. The set of numbers in the domain can be written as {1, 2, 3, 4, 5, . . .}. Notice that the domain is an infinite set of numbers, even though the table only displays the first five elements.

The range is $f(n)$ or (a_n), the output numbers that result from applying the rule $-(1 - 4^n)$. The set of numbers in the range, which is the sequence itself, can be written as {3, 7, 63, 255, 1023, . . .}. This is also an infinite set of numbers, even though the table only displays the first five elements.

SAMPLE ITEMS

1. Consider this pattern.



Which function represents the sequence that represents the pattern?

A. $a_n = (4)^{(n-1)}$

B. $a_n = (4)^{(a_n - 1)}$

C. $a_n = (a_n)(4)^{(n-1)}$

D. $a_n = (a_n)^4$

Correct Answer: A

2. Which function is modeled in this table?

x	$f(x)$
1	1000
2	800
3	640
4	512

A. 1,000(0.80)
B. 1,000(0.20)
C. $1{,}000(0.80)^x$
D. $1{,}000(0.20)^x$

Correct Answer: C

3. Which explicit formula describes the pattern in this table?

d	C
0	1
1	6
2	36
3	216

A. $C = 6d$
B. $C = d + 6$
C. $C = 6^d$
D. $C = d^6$

Correct Answer: C

4. If $f(12) = 100(0.50)^{12}$, which expression gives $f(x)$?

A. $f(x) = 12^x$
B. $f(x) = 100^x$
C. $f(x) = 100(x)^{12}$
D. $f(x) = 100(12)^x$

Correct Answer: D

Interpret Functions That Arise in Applications in Terms of the Context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior.

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

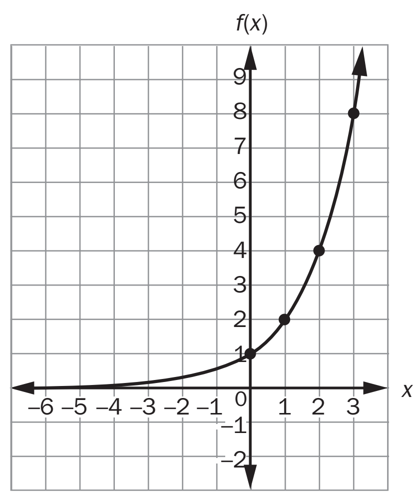
KEY IDEAS

1. By examining the graph of a function, many of its features are discovered. Features include domain and range; *x*- and *y*-intercepts; intervals where the function values are increasing, decreasing, positive, or negative; end behavior; relative maximum and minimum; and rates of change.

   Example:

   Consider the graph of $f(x) = 2^x$.

   Exponential Function
   $f(x) = 2^x$

   Some of its key features include the following:

   - Domain: All real numbers because there is a point on the graph for every possible *x*-value
   - Range: $y > 0$

- x-intercept: None
- y-intercept: It appears to intersect the y-axis at 1.
- Increasing: Always
- Decreasing: Never
- Positive: $f(x)$ is positive for all x-values.
- Negative: $f(x)$ is never negative.
- Rate of change: There is a variable rate of change, as the graph represents a curve. The rate of change for the interval $1 \leq x \leq 2$ is approximately 2 but is only 1 for $0 \leq x \leq 1$.
- Asymptote: $y = 0$

2. Other features of functions can be discovered through examining their tables of values. The intercepts may appear in a table of values. From the differences of $f(x)$-values over various intervals, we can tell if a function grows at a constant rate of change. Some intervals could have a different rate of change than other intervals.

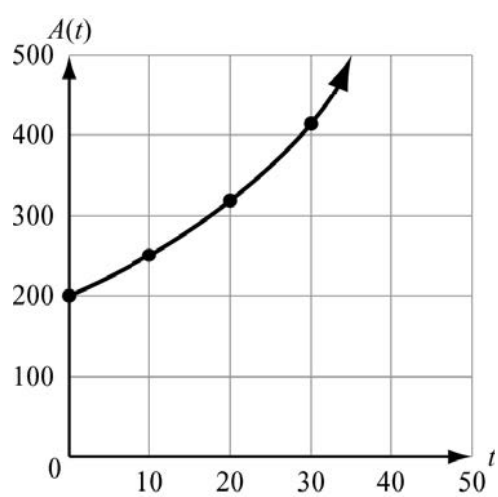
Important Tips

✍ You could begin exploration of a new function by generating a table of values using a variety of numbers from the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure to choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain.

✍ You cannot always find exact values from a graph. Always check your answers using the equation.

REVIEW EXAMPLES

1. The amount accumulated in a bank account over a time period t and based on an initial deposit of \$200 is found using the formula $A(t) = 200(1.025)^t$, $t \geq 0$. Time, t, is represented on the horizontal axis. The accumulated amount, $A(t)$, is represented on the vertical axis.



a. What are the intercepts of the function?
b. What is the domain of the function?
c. Why are all the t-values non-negative?
d. What is the range of the function?

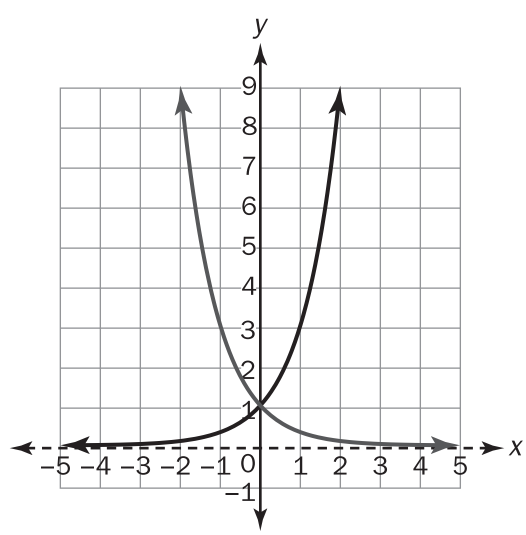
Solution:

a. There is no t-intercept because the bank account was never lower than $200. The function crosses the vertical axis at 200.
b. The domain is $t \geq 0$.
c. The t-values are all non-negative because they represent time, and time cannot be negative.
d. The range is $A(t) \geq 200$.

2. Consider two exponential functions, $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$. Compare the key features of the two functions.

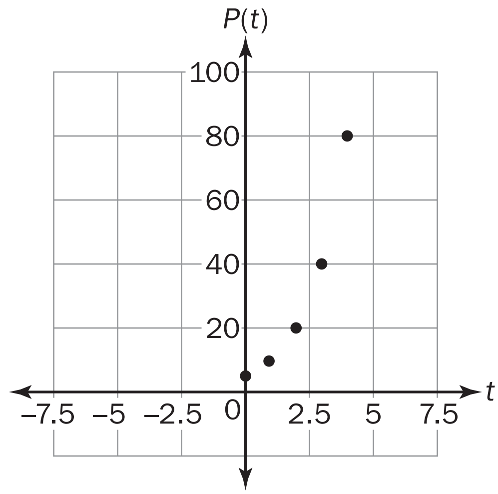
Solution:

Both have the same domain (all real numbers), range ($\{y|y > 0\}$), and y-intercept (0, 1). The function $f(x) = 3^x$ is an exponential growth function, while the function $g(x) = \left(\frac{1}{3}\right)^x$ is an exponential decay function.



SAMPLE ITEMS

1. A population of squirrels doubles every year. Initially, there were 5 squirrels. A biologist studying the squirrels created a function to model their population growth: $P(t) = 5(2^t)$, where t is the time in years. The graph of the function is shown.

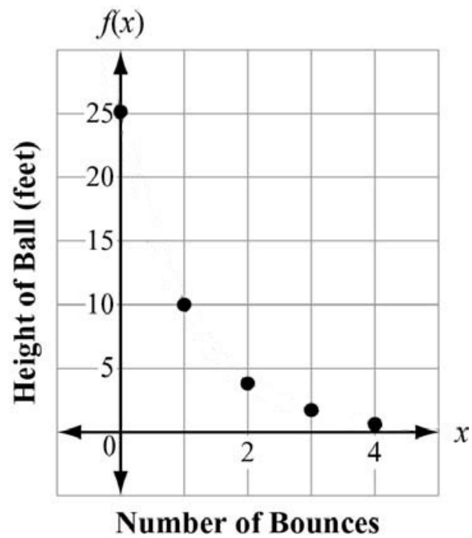


What is the range of the function?

A. any real number
B. any whole number greater than 0
C. any whole number greater than 5
D. any whole number greater than or equal to 5

Correct Answer: D

2. The function graphed on this coordinate grid shows $f(x)$, the height of a dropped ball in feet after its xth bounce.



On which bounce was the height of the ball 10 feet?

A. bounce 1
B. bounce 2
C. bounce 3
D. bounce 4

Correct Answer: A

Analyze Functions Using Different Representations

MGSE9-12.F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology.

MGSE9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.7e Graph exponential functions, showing intercepts and end behavior.

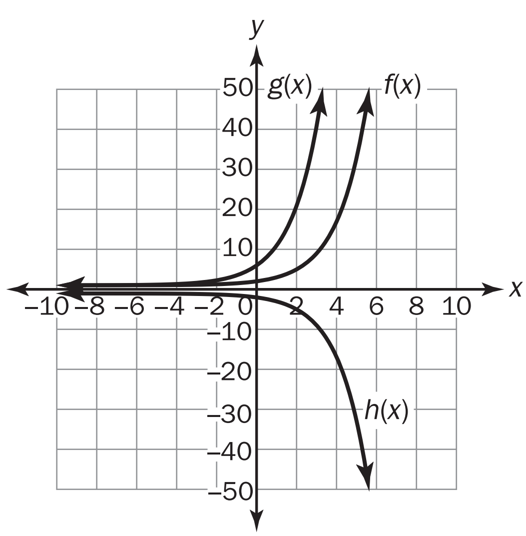
MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

KEY IDEA

1. The different ways of representing a function also apply to exponential functions. Exponential functions are built using powers. A power is the combination of a base with an exponent. For example, in the power 5^3, the base is 5 and the exponent is 3. A function with a power where the exponent is a variable is an exponential function. Exponential functions are of the form $f(x) = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$. In an exponential function, the base, b, is a constant and a is the coefficient.

   Example:

   Consider $f(x) = 2^x$, $g(x) = 5 \cdot 2^x$, and $h(x) = -1 \cdot 2^x$. For all three functions, $f(x)$, $g(x)$, and $h(x)$, the base is 2. The coefficient in $f(x)$ is 1, $g(x)$ is 5, and $h(x)$ is –1. The values of the coefficients cause the graphs to transform.



From the graphs, you can make the following observations:

- $f(x)$ appears to have a y-intercept at 1.
- $g(x)$ appears to have a y-intercept at 5.
- $h(x)$ appears to have a y-intercept at –1.
- For $f(x)$, as x increases, $f(x)$ increases, and as x decreases, $f(x)$ approaches 0.
- For $g(x)$, as x increases, $g(x)$ increases, and as x decreases, $g(x)$ approaches 0.
- For $h(x)$, as x increases, $h(x)$ decreases, and as x decreases, $h(x)$ approaches 0.
- None of the functions appear to have a constant rate of change.
- For $f(x)$, $g(x)$, and $h(x)$, the domain is all real numbers.
- For $f(x)$ and $g(x)$, the range is all real numbers greater than 0.
- For $h(x)$, the range is all real numbers less than 0.
- The asymptote for all of the functions is $y = 0$.

Now look at tables of values for these functions.

x	$f(x) = 2^x$
–3	$\frac{1}{8}$
–2	$\frac{1}{4}$
–1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16

x	$g(x) = 5 \cdot 2^x$
–3	$\frac{5}{8}$
–2	$\frac{5}{4}$
–1	$\frac{5}{2}$
0	5
1	10
2	20
3	40
4	80

x	$h(x) = -1 \cdot 2^x$
–3	$-\frac{1}{8}$
–2	$-\frac{1}{4}$
–1	$-\frac{1}{2}$
0	–1
1	–2
2	–4
3	–8
4	–16

The tables confirm all three functions have y-intercepts: $f(0) = 1$, $g(0) = 5$, and $h(0) = -1$. Although the tables do not show a constant rate of change for any of the functions, a rate of change can be determined on a specific interval by finding the change in the y-value divided by the change in the x-value for two distinct points on a graph.

Now let's represent $g(x) = 5 \cdot 2^x$ contextually. Let $g(x)$ be the population of bacteria where x is the number of days the bacteria population increases. The information provided in terms of bacteria and days is represented differently while using the key features used with tables and graphs.

- There were initially 5 bacteria prior to the population of bacteria increasing.
- The bacteria double each day.
- The bacteria population increases as the number of days increases.
- There is no maximum value.

- The minimum value is 5 bacteria.
- The number of bacteria will always range from 5 to infinity.

Important Tips

✍ Remember the elements of the domain and the values obtained by substituting them into the function rule form the coordinates of the points that lie on the graph of a function.

✍ Be familiar with important features of a function, such as intercepts, domain, range, minimums and maximums, end behavior, asymptotes, and periods of increasing and decreasing values.

✍ Also notice how the value of the functions change as they transform. The value of $g(–3)$ is 5 times more than $f(–3)$ and the value $h(–3)$ is the opposite value of $f(–3)$.

REVIEW EXAMPLE

1. Two quantities increase at exponential rates. This table shows the value of Quantity A, $f(x)$, after x years.

Quantity A

x	0	1	2	3	4
$f(x)$	100.00	150.00	225.00	337.50	506.25

This function represents the value of Quantity B, $g(x)$, after x years.

$$g(x) = 50(2)^x$$

Which quantity will be greater at the end of the fourth year and by how much?

Solution:

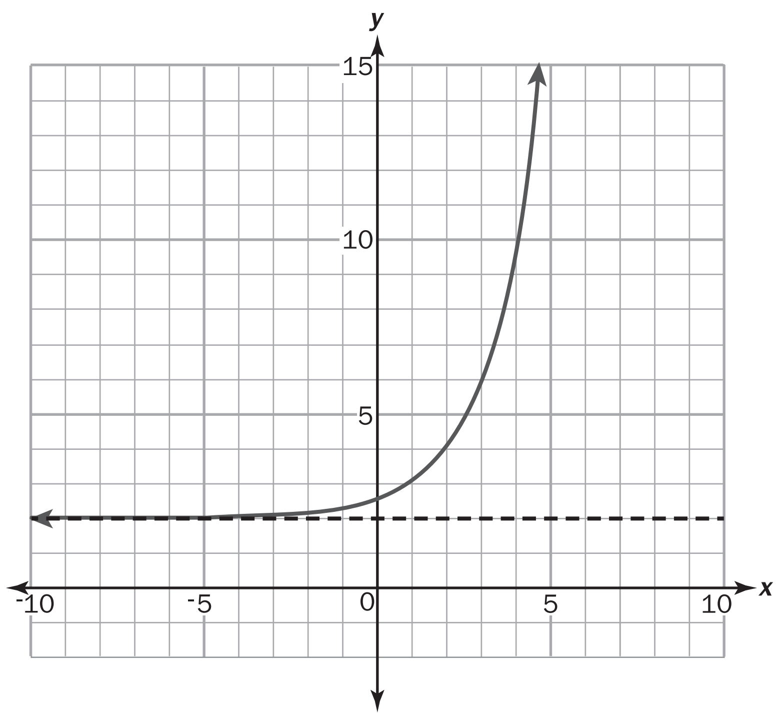
Find $g(4)$ for Quantity B:

$g(4) = 50(2)^4 = 50(16) = 800$

Quantity B will be 293.75 greater than Quantity A after 4 years.

SAMPLE ITEM

1. Look at the graph.



Which equation represents this graph?

A. $y = 2^{(x + 1)} - 2$
B. $y = 2^{(x - 1)} + 2$
C. $y = 2^{(x + 2)} - 1$
D. $y = 2^{(x - 2)} + 1$

Correct Answer: B

UNIT 5: COMPARING AND CONTRASTING FUNCTIONS

In this unit, students compare and contrast linear, quadratic, and exponential functions. They distinguish between situations that can be modeled by each type of function. They will build new functions from existing functions and interpret functions for specific situations.

Construct and Compare Linear, Quadratic, and Exponential Models and Solve Problems

MGSE9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.

> **MGSE9-12.F.LE.1a** Show that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals. (This can be shown by algebraic proof, with a table showing differences, or by calculating average rates of change over equal intervals.)
>
> **MGSE9-12.F.LE.1b.** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
>
> **MGSE9-12.F.LE.1c** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

MGSE9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

MGSE9-12.F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.

KEY IDEAS

1. Recognizing linear and exponential growth rates is key to modeling a quantitative relationship. The most common growth rates in nature are either linear or exponential. Linear growth happens when the dependent variable changes are the same for equal intervals of the independent variable. Exponential growth happens when the dependent variable changes at the same percent rate for equal intervals of the independent variable.

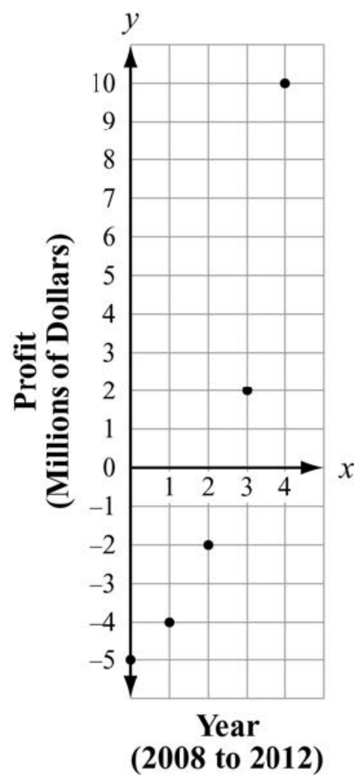
   Example:

   Given a table of values, look for a constant rate of change in the y, or $f(x)$, column. The table below shows a constant rate of change, namely, –2, in the $f(x)$ column for each unit change in the independent variable x. The table also shows the y-intercept of the relation. The function has a y-intercept of +1, the $f(0)$-value. These two pieces of information allow us to find a model for the relationship. When the change in $f(x)$ is constant, we use a linear model, $f(x) = ax + b$, where a represents the constant rate of change and b the y-intercept. For the given table, the a-value is –2, the constant change in the $f(x)$-values, and b is the $f(x)$-value of 1. The function is $f(x) = -2x + 1$. Using the linear model, we are looking for an explicit formula for the function.

x	$f(x)$	Change in $f(x)$
–2	5	—
–1	3	3 – 5 = –2
0	1	1 – 3 = –2
1	–1	–1 – 1 = –2
2	–3	–3 – (–1) = –2

Example:

Given the graph below, compare the coordinates of points to determine if there is either linear or exponential growth.



The points represent the profit/loss of a new company over its first 5 years, from 2008 to 2012. The company started out \$5,000,000 in debt. After 5 years, it had a profit of \$10,000,000. From the arrangement of the points, the pattern does not look linear. We can check by considering the coordinates of the points and using a table of values.

x	*y*	Change in *y*
0	–5,000,000	—
1	–4,000,000	–4,000,000 – (–5,000,000) = 1,000,000
2	–2,000,000	–2,000,000 – (–4,000,000) = 2,000,000
3	2,000,000	2,000,000 – (–2,000,000) = 4,000,000
4	10,000,000	10,000,000 – (2,000,000) = 8,000,000

The *y* changes are not constant for equal *x* intervals. However, the ratios of successive differences are equal. Therefore, it is confirmed that this is not linear.

$$\frac{2{,}000{,}000}{1{,}000{,}000} = \frac{4{,}000{,}000}{2{,}000{,}000} = \frac{2}{1} = 2$$

Having a constant percent for the growth rate for equal intervals indicates exponential growth. The relationship can be modeled using an exponential function. However, our example does not cross the *y*-axis at 1, or 1,000,000. Since the initial profit value was not \$1,000,000, the exponential function has been translated downward. The amount of the translation is \$6,000,000 because we are starting at year 0 instead of year 1. We model the company's growth as

$$P(x) = 1{,}000{,}000(2^x) - 6{,}000{,}000.$$

2. We can use our analysis tools to compare ***growth rates***. For example, it might be interesting to consider whether you would like your pay raises to be linear or exponential. Linear growth is characterized by a constant number. With a linear growth, a value grows by the same amount each time. Exponential growth is characterized by a percent which is called the growth rate.

   Example:

   Suppose you start work and earn $30,000 per year. After one year, you are given two choices for getting a raise: a) 2% per year or b) $600 plus a flat $15-per-year raise for each successive year. Which option is better? We can make a table with both options and see what happens.

Year	Yearly Pay	
	2%-per-year raise	$600 plus $15-per-year raise
1	$600.00	$600.00
2	$612.00	$615.00
3	$624.24	$630.00
4	$636.72	$645.00

   Looking at years 2 through 4, the $600 plus $15-per-year option seems better. However, look closely at the 2% column. Though the pay increases start out smaller each year, they are growing exponentially. For some year in the future, the 2%-per-year increase in salary will be more than the $15-per-year increase in salary. If you know the number of years you expect to work at the company, it will help determine which option is best.

3. Comparing functions helps us gain a better understanding of them. Let's take a look at linear, quadratic, and exponential functions.

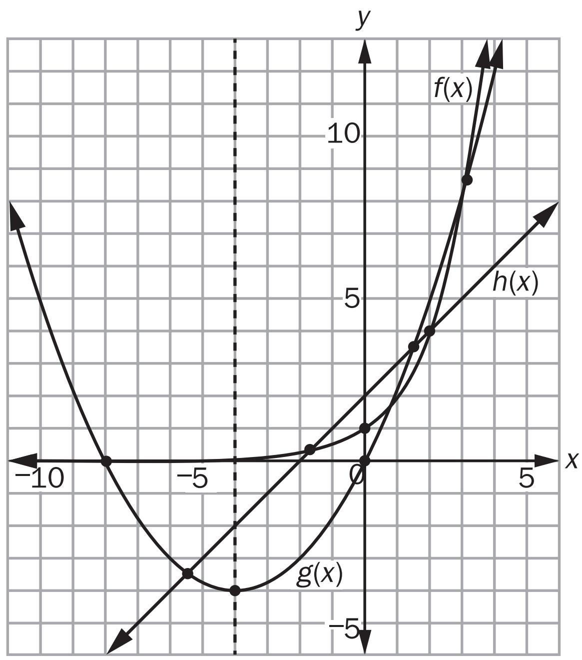
   Example:

   Consider the tables and the graphs of the functions.

Linear	
x	$h(x) = x + 2$
1	3
2	4
3	5
4	6
5	7
6	8

Quadratic	
x	$g(x) = 0.25x^2 + 2x$
1	2.25
2	5.00
3	8.25
4	12.00
5	16.25
6	21.00

Exponential	
x	$f(x) = 2^x$
1	2
2	4
3	8
4	16
5	32
6	64



In the table, $h(1)$ is greater than $g(1)$ which is greater than $f(1)$. The tables can be used to see that each function increases, and the graph provides a visual of the intervals each function decreases or increases. Notice how $f(x)$ increases at a higher rate than $g(x)$ and both increase at a higher rate than $h(x)$. The graph also displays that $g(x)$ is the only function that decreases and increases. The axis of symmetry passes through the vertex, which is the point where $g(x)$ begins to increase as x increases.

Important Tips

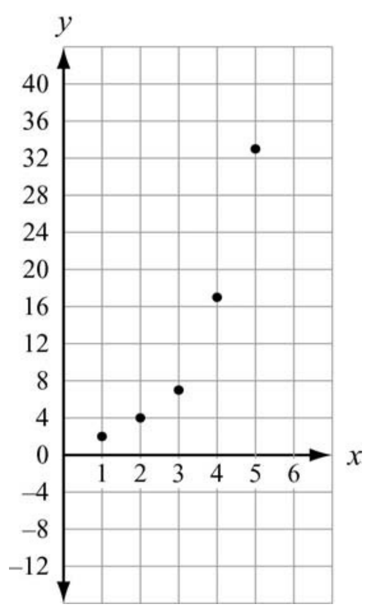
- ✍ Examine function values carefully.
- ✍ Remember that a linear function has a constant rate of change.
- ✍ Keep in mind that growth rates are modeled with exponential functions.
- ✍ Quadratic functions decrease and increase.
- ✍ Remember that the elements of the domain and the values obtained by substituting them into the function rule form the coordinates of the points that lie on the graph of a function. The domain and range can also be determined by examining the graph of a function, by looking for asymptotes on the graph of an exponential function, or by looking for endpoints or continuity for linear, quadratic, and exponential functions, or the vertex of a quadratic function.
- ✍ Be familiar with important features of a function such as intercepts, domain, range, minimum and maximums, end behavior, asymptotes, and periods of increasing and decreasing values.

REVIEW EXAMPLES

1. The swans on Elsworth Pond have been increasing in number each year. Felix has been keeping track, and so far he has counted 2, 4, 7, 17, and 33 swans each year for the past 5 years.

   a. Make a scatter plot of the swan populations.
   b. What type of model would be a better fit, linear or exponential? Explain your answer.
   c. How many swans should Felix expect next year if the trend continues? Explain your answer.

   Solution:

   a. 

   b. Exponential; the growth rate is not constant. The swan population appears to be nearly doubling every year.
   c. There could be about 64 swans next year. A function modeling the swan growth would be $P(x) = 2^x$, which would predict $P(6) = 2^6 = 64$.

2. Given the sequence 7, 10, 13, 16, . . .

   a. Does it appear to be linear or exponential?
   b. Determine a function to describe the sequence.
   c. What would the 20th term of the sequence be?

   Solution:

   a. Linear; the terms increase by a constant amount, 3.
   b. $f(x) = 3x + 4$. The growth rate is 3, and the first term is 4 more than 3 times 1.
   c. 64; $f(20) = 3(20) + 4 = 64$

3. This table shows that the value of $f(x) = 5x^2 + 4$ is greater than the value of $g(x) = 2^x$ over the interval [0, 8].

x	$f(x)$	$g(x)$
0	$5(0)^2 + 4 = 4$	$2^0 = 1$
2	$5(2)^2 + 4 = 24$	$2^2 = 4$
4	$5(4)^2 + 4 = 84$	$2^4 = 16$
6	$5(6)^2 + 4 = 184$	$2^6 = 64$
8	$5(8)^2 + 4 = 324$	$2^8 = 256$

As x increases, will the value of $f(x)$ always be greater than the value of $g(x)$? Explain how you know.

Solution:

For some value of x, the value of an exponential function will eventually exceed the value of a quadratic function. To demonstrate this, find the values of $f(x)$ and $g(x)$ for another value of x, such as $x = 10$.

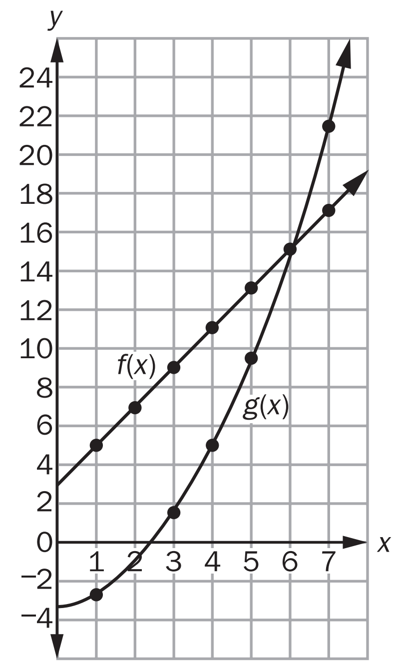
$$f(x) = 5(10)^2 + 4 = 504$$
$$g(x) = 2^{10} = 1{,}024$$

In fact, this means that for some value of x between 8 and 10, the value of $g(x)$ becomes greater than the value of $f(x)$ and remains greater for all subsequent values of x.

4. How does the growth rate of the function $f(x) = 2x + 3$ compare with $g(x) = 0.5x^2 - 3$? Use a graph to explain your answer.

Solution:

Graph $f(x)$ and $g(x)$ over the interval $x \geq 0$.



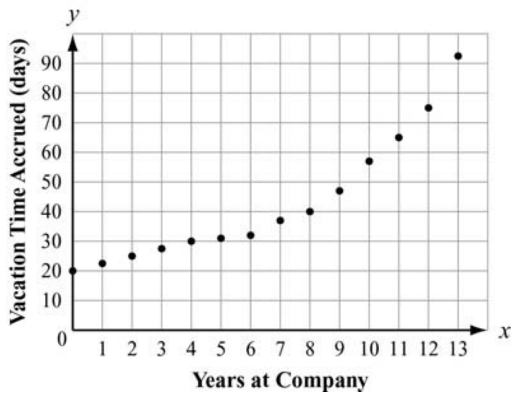
The graph of $f(x)$ increases at a constant rate because it is linear.

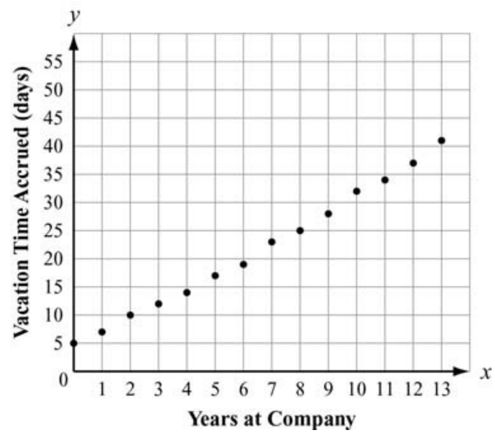
The graph of $g(x)$ increases at an increasing rate because it is quadratic.

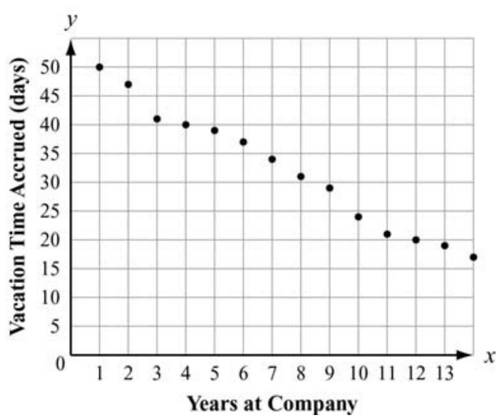
The graphs can be shown to intersect at (6, 15), and the value of $g(x)$ is greater than the value of $f(x)$ for $x > 6$.

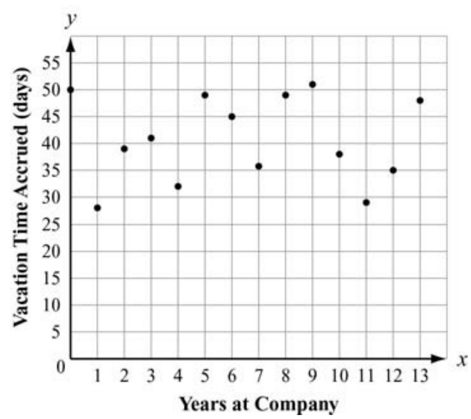
SAMPLE ITEMS

1. Which scatter plot BEST represents a model of linear growth?

A. 

B. 

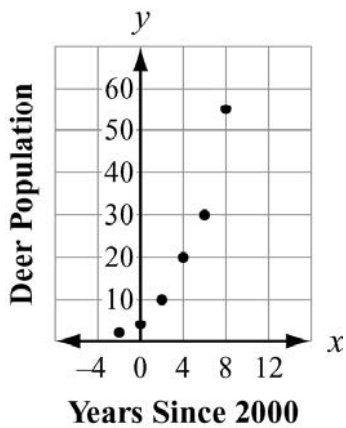
C. 

D. 

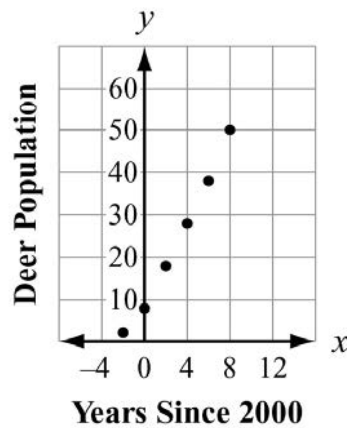
Correct Answer: B

2. Which scatter plot BEST represents a model of exponential growth?

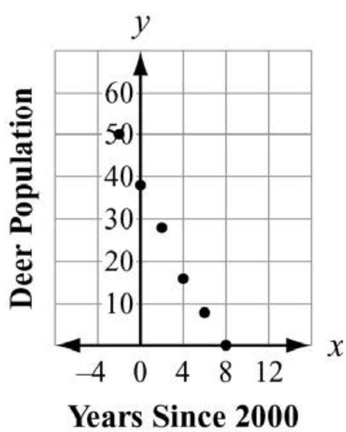
A.



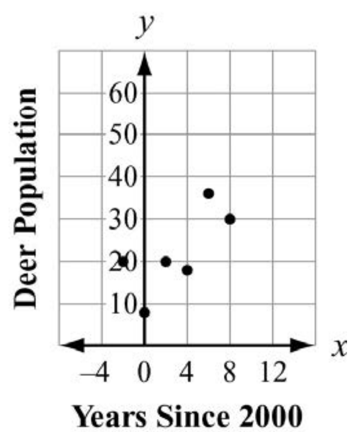
B.



C.



D.



Correct Answer: A

3. Which table represents an exponential function?

A.

x	0	1	2	3	4
y	5	6	7	8	9

B.

x	0	1	2	3	4
y	0	22	44	66	88

C.

x	0	1	2	3	4
y	5	13	21	29	37

D.

x	0	1	2	3	4
y	0	3	9	27	81

Correct Answer: D

4. A table of values is shown for $f(x)$ and $g(x)$.

x	$f(x)$
0	0
1	1
2	4
3	9
4	16
5	25

x	$g(x)$
0	–2
1	–1
2	1
3	5
4	13
5	29

Which statement compares the graphs of $f(x)$ and $g(x)$ over the interval [0, 5]?

A. The graph of $f(x)$ always exceeds the graph of $g(x)$ over the interval [0, 5].
B. The graph of $g(x)$ always exceeds the graph of $f(x)$ over the interval [0, 5].
C. The graph of $g(x)$ exceeds the graph of $f(x)$ over the interval [0, 4], the graphs intersect at a point between 4 and 5, and then the graph of $f(x)$ exceeds the graph of $g(x)$.
D. The graph of $f(x)$ exceeds the graph of $g(x)$ over the interval [0, 4], the graphs intersect at a point between 4 and 5, and then the graph of $g(x)$ exceeds the graph of $f(x)$.

Correct Answer: D

5. Which statement is true about the graphs of exponential functions?

A. The graphs of exponential functions never exceed the graphs of linear and quadratic functions.
B. The graphs of exponential functions always exceed the graphs of linear and quadratic functions.
C. The graphs of exponential functions eventually exceed the graphs of linear and quadratic functions.
D. The graphs of exponential functions eventually exceed the graphs of linear functions but not quadratic functions.

Correct Answer: C

6. Which statement BEST describes the comparison of the function values for $f(x)$ and $g(x)$?

x	$f(x)$	$g(x)$
0	0	–10
1	2	–9
2	4	–6
3	6	–1
4	8	6

A. The values of $f(x)$ will always exceed the values of $g(x)$.
B. The values of $g(x)$ will always exceed the values of $f(x)$.
C. The values of $f(x)$ exceed the values of $g(x)$ over the interval [0, 5].
D. The values of $g(x)$ begin to exceed the values of $f(x)$ within the interval [4, 5].

Correct Answer: D

Interpret Expressions for Functions in Terms of the Situation They Model

MGSE9-12.F.LE.5 Interpret the parameters in a linear ($f(x) = mx + b$) and exponential ($f(x) = a \cdot dx$) function in terms of a context. (In the functions above, "m" and "b" are the parameters of the linear function, and "a" and "d" are the parameters of the exponential function.) In context, students should describe what these parameters mean in terms of change and starting value.

KEY IDEAS

1. A ***parameter*** is the independent variable or variables in a system of equations with more than one dependent variable. Though parameters may be expressed as letters when a relationship is generalized, they are not variables. A parameter as a constant term generally affects the intercepts of a function. If the parameter is a coefficient, in general it will affect the rate of change. Below are several examples of specific parameters.

Equation	Parameter(s)
$y = 3x + 5$	coefficient 3, constant 5
$f(x) = \frac{9}{5}x + 32$	coefficient $\frac{9}{5}$, constant 32
$v(t) = v_0 + at$	coefficient a, constant v_0
$y = mx + b$	coefficient m, constant b

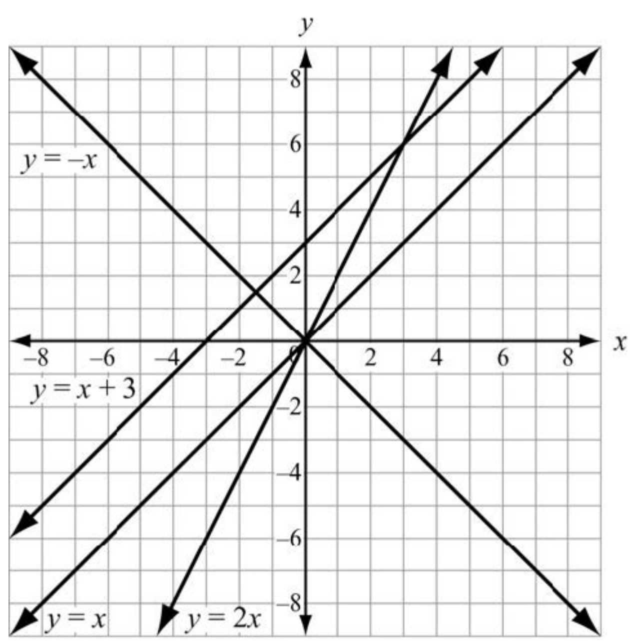
We can look at the effect of parameters on a linear function.

Example:

Consider the lines $y = x$, $y = 2x$, $y = -x$, and $y = x + 3$. The coefficients of x are parameters. The +3 in the last equation is a parameter. We can make one table for all four lines and then compare their graphs.

x	***y***			
	$y = x$	$y = 2x$	$y = -x$	$y = x + 3$
–3	–3	–6	3	0
–2	–2	–4	2	1
–1	–1	–2	1	2
0	0	0	0	3
1	1	2	–1	4
2	2	4	–2	5
3	3	6	–3	6
4	4	8	–4	7

The four linear graphs show the effects of the parameters.



- Only $y = x + 3$ has a different y-intercept. The +3 translated the $y = x$ graph up 3 units.
- Both $y = x$ and $y = x + 3$ have the same slope (rate of change). The coefficients of the x-terms are both 1.
- The lines $y = -x$ and $y = 2x$ have different slopes than $y = x$. The coefficients of the x-terms, −1 and 2, affect the slopes of the lines.
- The line $y = -x$ is the reflection of $y = x$ over the x-axis. It is the only line with a negative slope.
- The rate of change of $y = 2x$ is twice that of $y = x$.

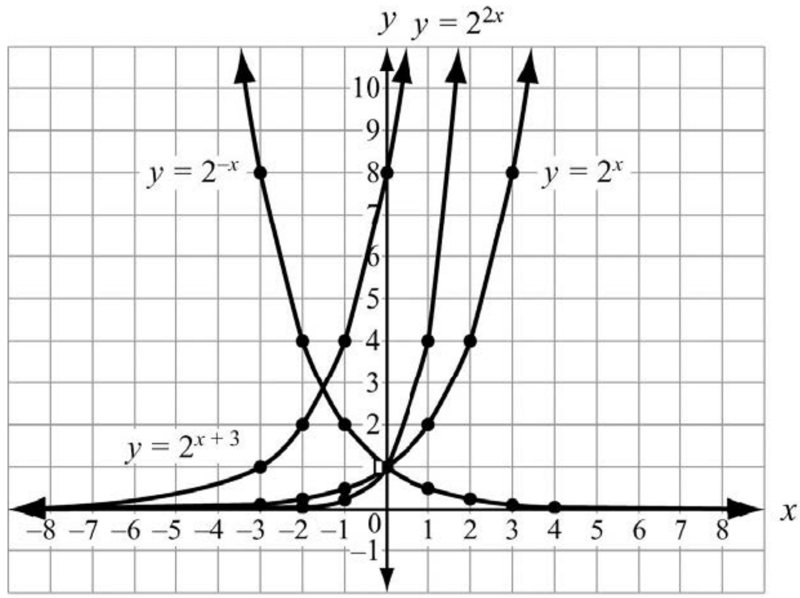
2. We can look at the effect of parameters on an exponential function, in particular, when applied to the independent variable, not the base.

Example:

Consider the exponential curves $y = 2^x$, $y = 2^{-x}$, $y = 2^{2x}$, and $y = 2^{x+3}$. The coefficients of the exponent x are parameters. The +3 applied to the exponent x in the last equation is a parameter. We can make one table for all four exponentials and then compare the effects.

- $y = 2^{-x}$ is a mirror image of $y = 2^x$ with the y-axis as mirror. It has the same y-intercept.
- $y = 2^{2x}$ has the same y-intercept as $y = 2^x$ but rises much more steeply.
- $y = 2^{x+3}$ is the $y = 2^x$ curve translated 3 units to the left.

x	y			
	$y = 2^x$	$y = 2^{-x}$	$y = 2^{2x}$	$y = 2^{x+3}$
–3	$\frac{1}{8}$	8	$\frac{1}{64}$	1
–2	$\frac{1}{4}$	4	$\frac{1}{16}$	2
–1	$\frac{1}{2}$	2	$\frac{1}{4}$	4
0	1	1	1	8
1	2	$\frac{1}{2}$	4	16
2	4	$\frac{1}{4}$	16	32
3	8	$\frac{1}{8}$	64	64
4	16	$\frac{1}{16}$	256	128



y
$y = 2^{2x}$
$y = 2^{-x}$
$y = 2^{x}$
$y = 2^{x+3}$
x
10
9
8
7
6
5
4
3
2
0
–1
–8 –7 –6 –5 –4 –3 –2 –1
1 2 3 4 5 6 7 8

3. Parameters show up in equations when there is a parent function. Parameters affect the shape and position of the parent function. When we determine a function that models a specific set of data, we are often called upon to find the parent function's parameters.

Example:

Katherine has heard that you can estimate the outside temperature from the number of times a cricket chirps. It turns out that the warmer it is outside, the more a cricket will chirp. She has these three pieces of information:

- A cricket chirps 76 times a minute at 56° (76, 56).
- A cricket chirps 212 times per minute at 90° (212, 90).
- The relationship is linear.

Estimate the function.

Solution:

The basic linear model or parent function is $f(x) = mx + b$, where m is the slope of the line and b is the y-intercept.

So, the slope, or rate of change, is one of our parameters. First we will determine the constant rate of change, called the slope, m.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{90 - 56}{212 - 76} = \frac{34}{136} = \frac{1}{4}$$

Since we now know that $f(x) = \frac{1}{4}x + b$, we can substitute in one of our ordered pairs to determine b.

$$\begin{aligned} T(76) = 56, \text{ so } \frac{1}{4}(76) + b &= 56 \\ 19 + b &= 56 \\ 19 + b - 19 &= 56 - 19 \\ b &= 37 \end{aligned}$$

Our parameters are $m = \frac{1}{4}$ and $b = 37$.

Our function for the temperature is $T(c) = \frac{1}{4}c + 37$.

REVIEW EXAMPLES

1. Alice finds that her flower bulbs multiply each year. She started with just 24 tulip plants. After one year she had 72 plants. Two years later she had 120. Find a linear function to model the growth of Alice's bulbs.

Year	0	1	2	3	4
Flower Bulbs	24	72	120	168	216

Solution:

The data points are (0, 24), (1, 72), and (2, 120). The linear model is $B(p) = m(p) + b$.

We know $b = 24$ because $B(0) = 24$ and $B(0)$ gives the vertical intercept.

Find m: $m = \dfrac{120 - 72}{2 - 1} = \dfrac{48}{1} = 48$.

The parameters are $m = 48$ and $b = 24$.

The function modeling the growth of the bulbs is $B(p) = 48p + 24$.

2. Suppose Alice discovers she counted wrong the second year and she actually had 216 tulip plants. She realizes the growth is not linear because the rate of change was not the same. She must use an exponential model for the growth of her tulip bulbs. Find the exponential function to model the growth.

Solution:

We now have the points (0, 24), (1, 72), and (2, 216). We use a parent exponential model:

$$B(p) = a(b^x).$$

In the exponential model, the parameter a would be the initial number. So, $a = 24$. To find the base b, we substitute a coordinate pair into the parent function.

$$B(1) = 72, \text{ so } 24(b^1) = 72,\ b^1 = \frac{72}{24} = 3, \text{ so } b = 3.$$

Now we have the parameter and the base. The exponential model for Alice's bulbs would be

$$B(p) = 24(3^p).$$

SAMPLE ITEMS

1. **If the parent function is $f(x) = mx + b$, what is the value of the parameter m for the line passing through the points (–2, 7) and (4, 3)?**

   A. -9

   B. $-\frac{3}{2}$

   C. -2

   D. $-\frac{2}{3}$

Correct Answer: D

2. **Consider this function for cell duplication. The cells duplicate every minute.**

$$f(x) = 75(2)^x$$

   Describe the parameters of this function.

   Solution:

   Seventy-five is the initial number of cells. The 2 indicates that the number of cells doubles every minute.

Build New Functions from Existing Functions

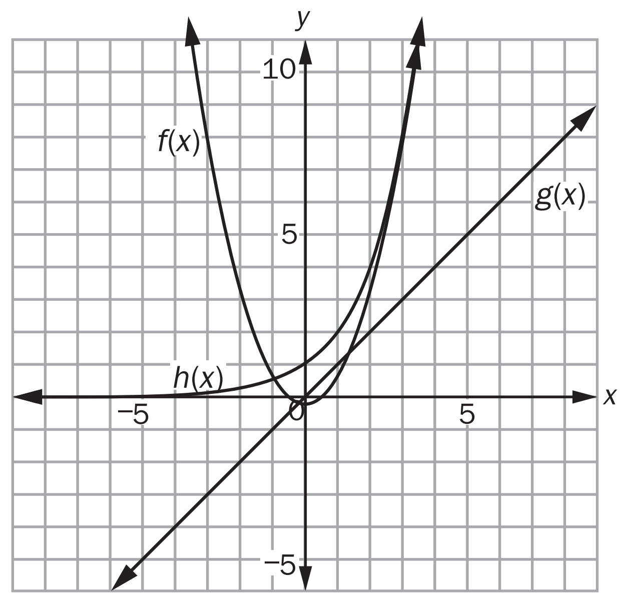
MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

KEY IDEAS

1. A ***parent function*** is the basic function from which all the other functions in a function family are modeled.

   - For the quadratic function family, the parent function is $f(x) = x^2$.
   - For the linear function family, the parent function is $g(x) = x$.
   - For the exponential function family, the parent function is $h(x) = 2^x$

Translations shift a function's graph and can change how different functions compare.



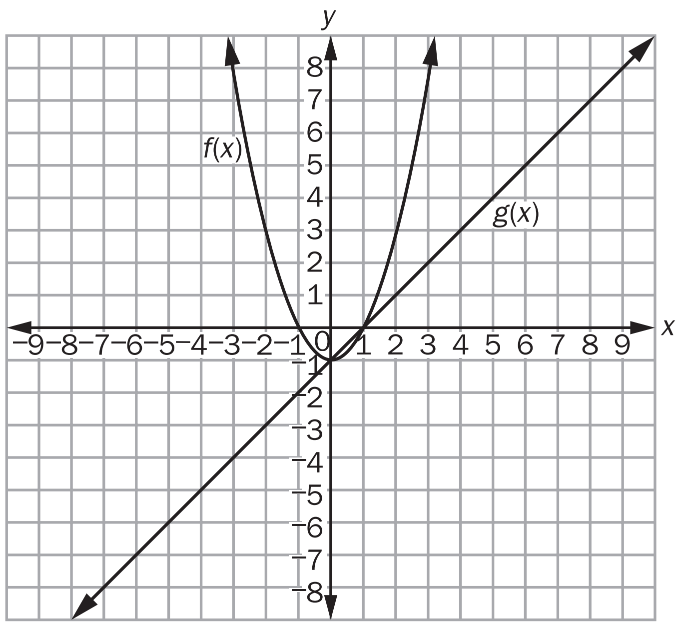
By looking at the graphs of the parent functions and comparing key features, notice the x- and y-intercepts of quadratic function f and the linear function g are both 0. This means that any transformations of these parent functions will shift the orientation of the graphs of the functions from the origin. Notice this is the vertex of the quadratic function. The exponential function, h, has a y-intercept at (0, 1) and an asymptote at $y = 0$.

Important Tip

✍ Remember that when you change $f(x)$ to $f(x + k)$, move the graph to the **left** when k is positive and to the **right** when k is negative. This may seem different from what you would expect, so be sure to understand why this occurs in order to apply the shift or translation correctly.

REVIEW EXAMPLES

1. Look at the graphs of the function $f(x) = x^2 + 1$ and $g(x) = x - 1$.



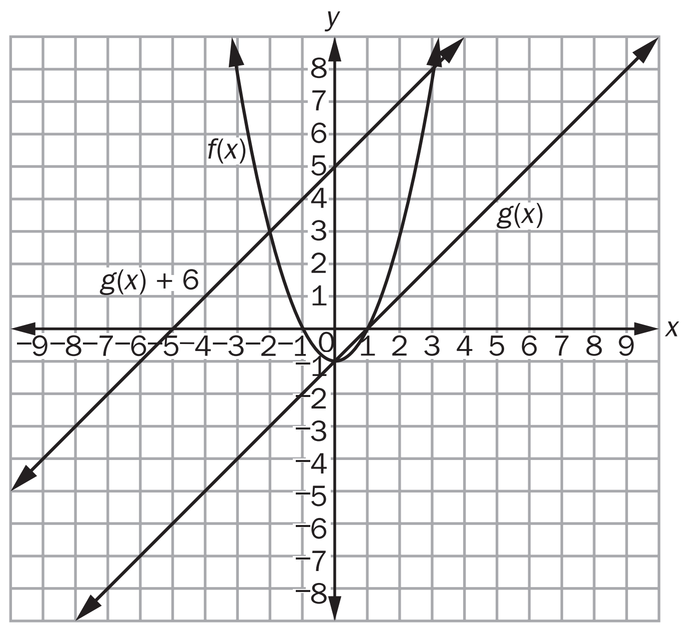
What transformation makes $g(x) \geq f(x)$ only for the interval $-2 \leq x \leq 3$?

Solution:

Since $g(x)$ passes through the points $(-2, -3)$ and $(3, -3)$, both points need to shift 6 units up, so that $g(x) \geq f(x)$ for $-2 \leq x \leq 2$. This transformation of $g(x)$ can be represented by $g(x) + 6$.

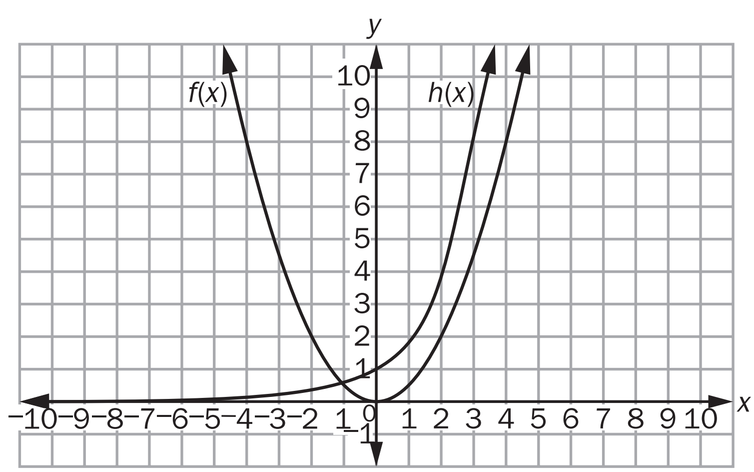
$$\begin{aligned} g(x) + 6 &= (x - 1) + 6 \\ &= x - 5 \end{aligned}$$

Let's take a look at the graphs of the functions.



Notice all points on the graph of $g(x)$ shifted up 6 units.

2. Look at the graphs of $f(x) = \left(\frac{1}{2}\right)x^2$ and $h(x) = 2^x$.



How does $f(x)$ need to transform so that there are some values for $x \geq 0$ so that $f(x) > h(x)$?

Solution:

Since the graph of $f(x)$ needs to horizontally shrink for some values for $x \geq 0$ so that $f(x) > h(x)$, then the transformation must be an integer k large enough so that $f(kx) > h(x)$ for $x \geq 0$. Let's try $k = 4$.

$$f(4x) = \left(\frac{1}{2}\right)(4x)^2$$

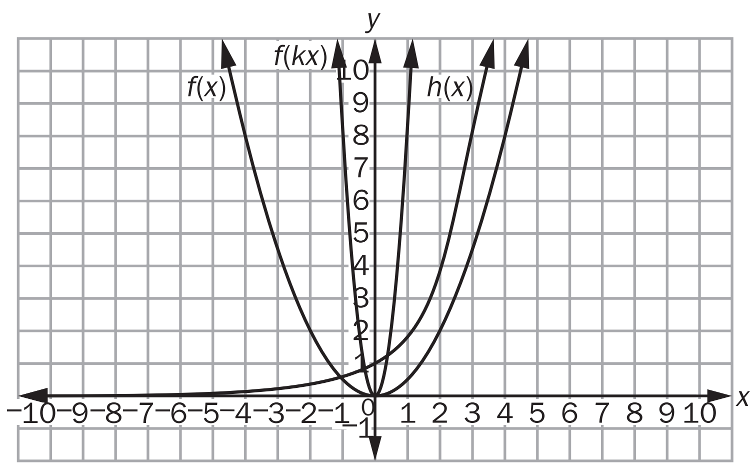
$$= \left(\frac{1}{2}\right)(16)(x)^2$$

$$= 8(x)^2$$

Let's try $x = 2$ and see if $f(kx) > h(x)$.

$f(4 \cdot 2) = 8(2)^2 = 8(4) = 32$ and $h(2) = 2^2 = 4$. So, $f(kx) > h(x)$.

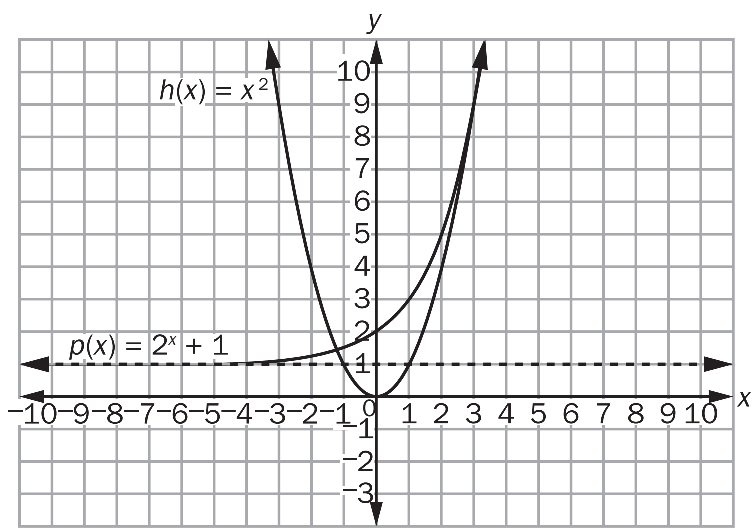
Let's revisit the graph with $f(kx)$ shown.



The graph also shows more values so that $f(kx) > h(x)$.

SAMPLE ITEMS

1. Look at the graph of the functions $h(x)$ and $p(x)$.



Which transformations of $h(x)$ and $p(x)$ translate each function so both pass through the point (0, 1)?

A. $h(x - 1) = (x - 1)^2$ and $p(x + 1) = 2^{(x + 1)} + 1$
B. $h(x + 1) = (x + 1)^2$ and $p(x - 1) = 2^{(x - 1)} + 1$
C. $h(x) - 1 = x^2 - 1$ and $p(x) + 1 = 2^x + 1$
D. $h(x) + 1 = x^2 + 1$ and $p(x) - 1 = 2^x - 1$

Correct Answer: D

2. Look at the functions *f*(*x*) and *g*(*x*).

$$f(x) = x^2$$
$$g(x) = 2^x + 3$$

Which transformation of *f*(*x*) makes *f*(*x*) < *g*(*x*)?

A. $f(-x)$

B. $-f(x)$

C. $\frac{1}{2}f(x)$

D. $2f(x)$

Correct Answer: B

Understand the Concept of a Function and Use Function Notation

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, (i.e., each input value maps to exactly one output value). If f is a function, x is the input (an element of the domain), and $f(x)$ is the output (an element of the range). Graphically, the graph is $y = f(x)$.

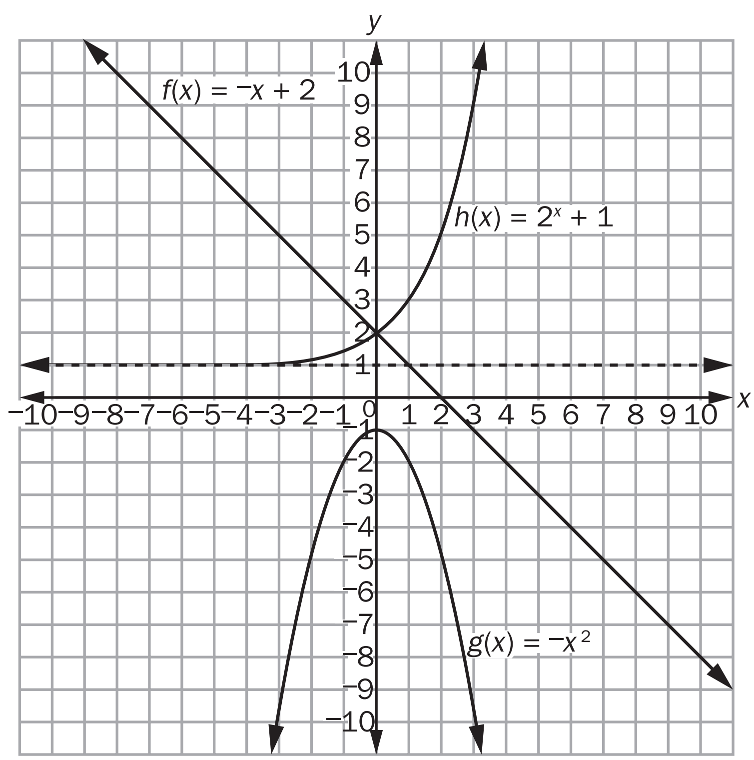
MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

KEY IDEAS

1. The ***domain*** and ***range*** of different functions can be identified by key features. In some cases, the domain and range can be determined by the context a function represents.

   Look at the three functions and their graphs.

   $f(x) = -x + 2$ $\quad$ $g(x) = -x^2$ $\quad$ $h(x) = 2^x + 1$



   All three functions have the same domain: the set of all real numbers. All three functions have exactly one unique output for every input. However, not all three functions have the same range.

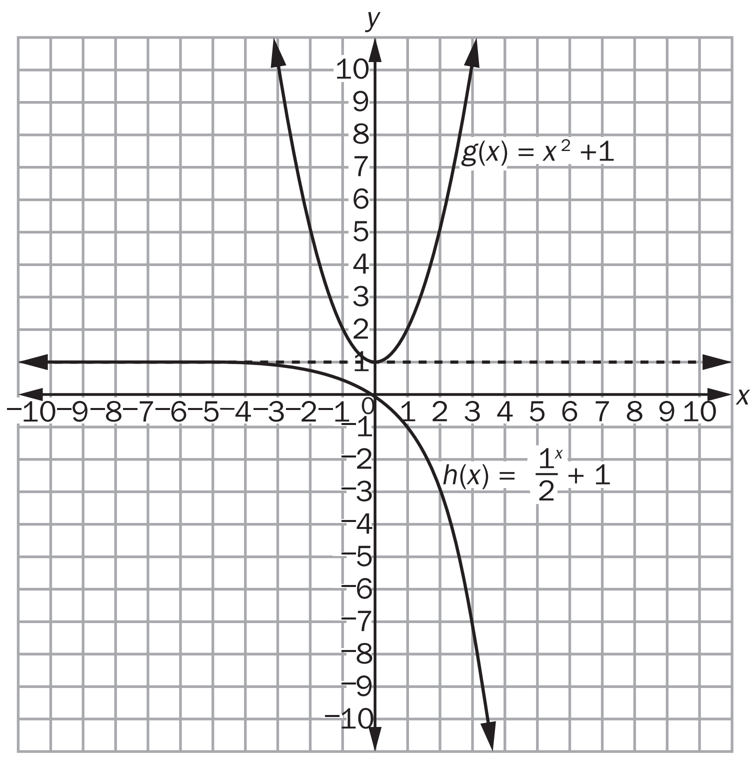
   - The range of $f(x) = -x + 2$ is all real numbers. This function has no maximum or minimum.
   - The range of $g(x) = -x^2$ is $g(x) \leq -1$. Notice the graph has a maximum of -1. This means there will be no value of y greater than -1.

- The range of $h(x) = 2^x + 1$ is $h(x) > 1$. This function has an asymptote at $y = 1$. This means the function will get closer and closer to 1 but will never equal 1. Since the graph is increasing, the range will be all values greater than 1.

2. The range on an infinite interval for exponential functions and quadratic functions can change depending on the key features of the function.

   Look at these functions and their graphs.

   $$g(x) = x^2 + 1 \qquad h(x) = \frac{1^x}{2} + 1$$



- The range of $g(x)$ is now $g(x) \geq 1$. Notice the graph shows a minimum of 1 which means there is no value of $y < 1$.
- The range of $h(x)$ is now $h(x) < 1$. The asymptote is still $y = 1$, so $h(x)$ will never equal 1. The growth factor has changed to a factor of less than 1, making this an exponential decay function.

3. The domain and range can be closed in contextual situations, which may not make sense for the full range of values allowed by the mathematical models.

   Examples:

   A ball being thrown from a height of 6 feet travels 10 yards.

   An exponential model of decay of a population of 1,000 bacteria begins at time $t = 0$, so the domain does not include negative time values. It is limited to values of $t \geq 0$. The range in this context is limited to $0 \leq y \leq 1,000$.

REVIEW EXAMPLES

1. A manufacturer keeps track of her monthly costs by using a “cost function” that assigns a total cost for a given number of manufactured items, x. The function is $C(x) = 5{,}000 + 1.3x$.

   a. What is the reasonable domain of the function?
   b. What is the cost of 2,000 items?

   Solution:

   a. Since x represents a number of manufactured items, it cannot be negative, nor can a fraction of an item be manufactured. Therefore, the domain can only include values that are whole numbers.
   b. Substitute 2,000 for x: $C(2{,}000) = 5{,}000 + 1.3(2{,}000) = \$7{,}600$.

2. As the input x increases by a factor of 3, the output $g(x)$ doubles. What type of function fits this situation?

   Solution:

   A linear function increases at a constant rate, not as a multiple, so this is not a linear function. Though the rate increases by a constant factor, meaning it is exponential rather than quadratic, the increase would follow an exponential model with a base of 2 and an exponent of $\frac{x}{3}$.

SAMPLE ITEMS

1. Which function is modeled in this table?

x	$f(x)$
1	8
2	40
3	200
4	1,000

A. $f(x) = x + 7$

B. $f(x) = 5x + 8$

C. $f(x) = (8)^x$

D. $f(x) = \frac{8}{5}(5)^x$

Correct Answer: D

2. If $f(12) = 4(12) - 20$, which function gives $f(x)$?

A. $f(x) = 4x^2 - 20$
B. $f(x) = 4^x - 20$
C. $f(x) = 4x - 20$
D. $f(x) = 4x^2 + 12x - 20$

Correct Answer: C

3. Which function has a range of $f(x) \le \frac{3}{4}$?

A. $f(x) = \frac{3}{4}x + 5$

B. $f(x) = -x^2 + \frac{3}{4}$

C. $f(x) = x^2 - \frac{3}{4}$

D. $f(x) = \frac{3}{4} - 5x$

Correct Answer: B

Interpret Functions That Arise in Applications in Terms of the Context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including intercepts; interval where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; and end behavior.

MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

KEY IDEAS

1. By examining the graph of a function, many of its features are discovered. Features include domain and range, x- and y-intercepts, intervals where the function values are increasing or decreasing, positive or negative, any symmetry in the graph, relative maxima and minima, and rates of change.
    - When a function models a context, such as a real-world scenario, consider the impact of the context upon the function. Behaviors and key features of the function provide information about the relationship modeled between the input and output values. Similarly, the context may place constraints and limitations upon the model.
2. The domain represents the input values of a functional model. Mathematical functions are often **continuous**, which means that the input and output values do not have any gaps. The context being modeled, on the other hand, may be **discrete**. A discrete function is defined only for a set of values that can be listed.
    - The discrete set may contain infinitely many elements, such as the whole numbers or the integers. Though you cannot list all of the elements, you can describe every element of the set in order, so they are considered discrete sets.
    - The rational numbers, on the other hand, are not a discrete set because any two rational numbers contain another rational number between them. (In fact, they contain infinitely many.) The same applies to the set of all real numbers.
3. The range represents the output values of the functional model. Like the domain, it too may be limited by the context to only certain types of numerical values. Models of an animal population, for example, would have a range of only the whole numbers, since the model represents the total number of animals within the population.
4. When viewing a graph of a functional model, it is important to identify any undefined regions of the function's domain and range.
    - Remember that a graph may represent only a portion of the mathematical function being used to model the situation. The function may be defined beyond what is shown on the graph.

- Discontinuities and undefined values within a functional model often represent important values within the context. Ask yourself how these features relate to the scenario described, as they may provide clues to using the model to understand the phenomena being modeled.

5. Tables may suggest a trend in the data or regions that are undefined, such as when the trend approaches an asymptote. Graphing a table of data often makes the type of model clearer.
6. If it is not apparent which type of functional model a graph or table represents, the average rate of change over different intervals of the function can give you a clue.
   - If any two intervals have the same average rate of change, the data is modeled by a linear function, of the form $y = mx + b$.
   - Data that increases and then changes to a decreasing rate, or decreases and then changes to an increasing rate, may be modeled by a quadratic function, of the form $y = ax^2 + bx + c$.
   - Data that is always increasing or always decreasing, but has a changing rate of change, may be modeled by an exponential function, of the form $y = ab^x$.

Important Tips

✍ One could begin exploration of a new function by generating a table of values using a variety of numbers for the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure to choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain. Other methods could be to analyze key features of the function from an equation.

✍ You cannot always find exact values from a graph. If provided with an equation, always check your answers using the equation.

✍ Be familiar with important features of a function such as intercepts, domain, range, minimum and maximums, end behavior, asymptotes, and periods of increasing and decreasing values. These features often give you key information about the context of a real-world scenario.

REVIEW EXAMPLES

1. Roger is washing cars for people in his neighborhood. He bought cleaning supplies with his own money before he began washing cars. He charges a flat fee of $15 for each car washed. Roger's total amount of profit, y, in dollars, for washing x cars can be modeled by the function $y = 15x - 40$.

   a. What is the domain of this function?

   b. What does the y-intercept of this function represent?

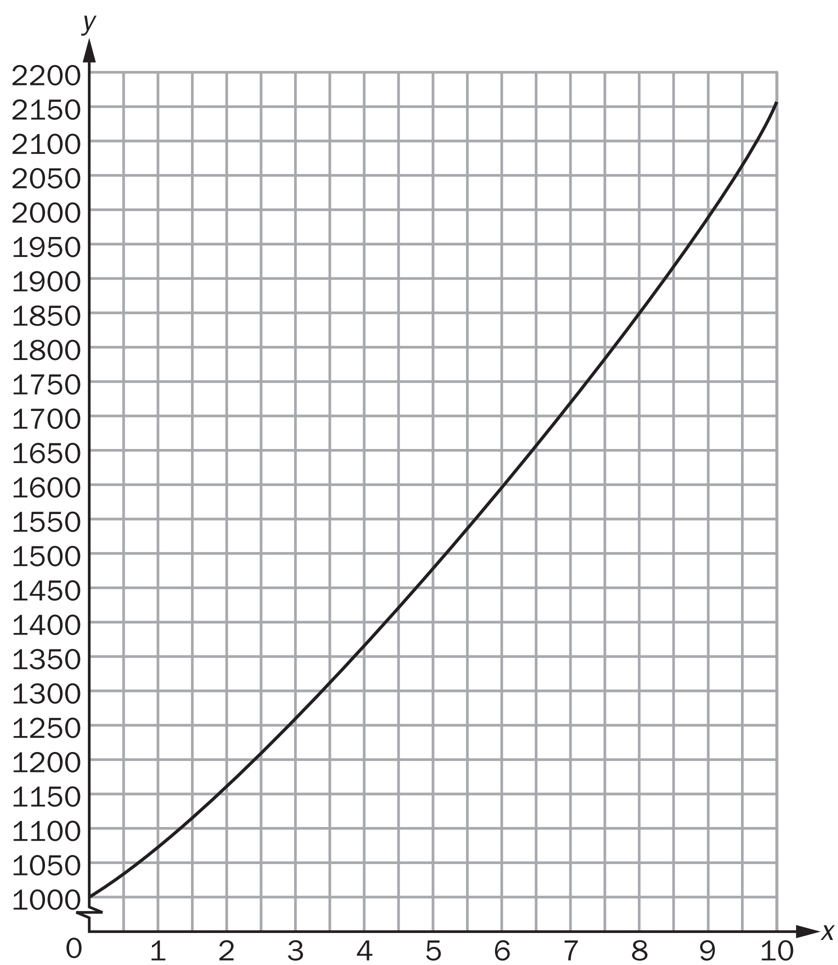
   Solution:

   a. The input values of this function represent the number of cars washed. It is impossible for Roger to wash a negative number of cars, so the domain is limited to $x \geq 0$. Since Roger is charging a flat fee per car washed, the context only applies to situations where he washes whole cars, so the domain represents only whole numbers of cars. The real numbers and rational numbers do not describe the possible domain, since no partial cars are being washed.

b. The y-intercept is at –40. If Roger washes 0 cars, his total profit will be a loss of \$40. This represents the amount of his own money that he spent on cleaning supplies before he began washing cars.

2. Miranda has an investment that earns 8% interest each year. She calculates that over the first 5 years, her \$1,000 investment will earn an average of approximately \$94 per year. At this rate, she thinks it will take more than 10 years to double her money.

   The graph shows the function modeling her investment, $V(t) = 1{,}000(1.08)^t$, where t represents the time in years.



   a. Approximately how many years does it actually take for Miranda to double her initial investment?

   b. Explain why Miranda's estimate was incorrect.

Solution:

a. Since the initial investment is \$1,000, her investment will have doubled when the total value of the investment is \$2,000, as indicated on the *y*-axis of the graph. This value is obtained on the function at an input value of 9 years.

b. The average rate of change for the first 5 years is about \$94 per year, but Miranda's estimate was incorrect because she did not realize the growth was exponential. The increase of 8% every year, indicated by the base of 1.08 in the function, causes the amount earned each year to change. Her investment is worth approximately \$2,000, twice the original \$1,000 investment, after only 9 years. During this time, it has earned an average annual return of $\frac{\$2000}{9} \approx \$111$.

SAMPLE ITEMS

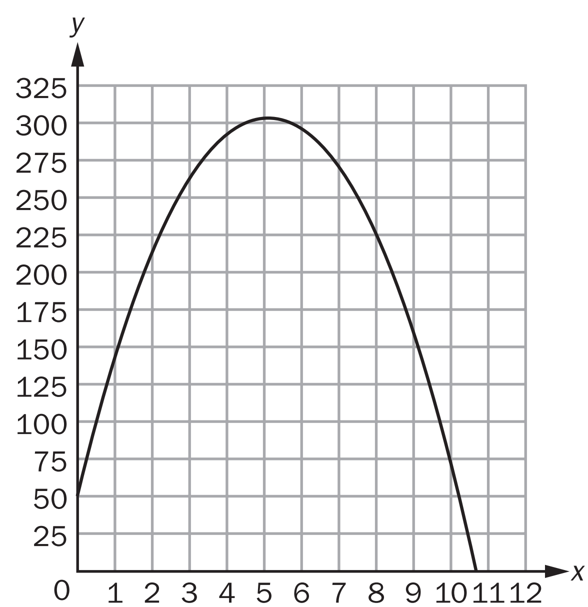
1. **A sample of 1,000 bacteria becomes infected with a virus. Each day, one-fourth of the bacteria sample dies due to the virus. A biologist studying the bacteria models the population of the bacteria with the function $P(t) = 1{,}000(0.75)^t$, where *t* is the time, in days.**

   What is the range of this function in this context?

   A. any real number such that $t \geq 0$
   B. any whole number such that $t \geq 0$
   C. any real number such that $0 \leq P(t) \leq 1{,}000$
   D. any whole number such that $0 \leq P(t) \leq 1{,}000$

Correct Answer: D

2. **The graph shows the height, *y*, in meters, of a rocket above sea level in terms of the time, *t*, in seconds, since it was launched. The rocket landed at sea level.**



   What does the *x*-intercept represent in this situation?

   A. the height from which the rocket was launched
   B. the time it took the rocket to return to the ground
   C. the total distance the rocket flew while it was in flight
   D. the time it took the rocket to reach the highest point in its flight

Correct Answer: B

UNIT 6: DESCRIBING DATA

In this unit, students will learn informative ways to display both categorical and quantitative data. They will learn ways of interpreting those displays and pitfalls to avoid when presented with data. Students will learn how to determine the mean absolute deviation. Among the methods they will study are two-way frequency charts for categorical data and lines of best fit for quantitative data. Measures of central tendency will be revisited along with measures of spread.

Summarize, Represent, and Interpret Data on a Single Count or Measurable Variable

MGSE9-12.S.ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

MGSE9-12.S.ID.2 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, mean absolute deviation) of two or more different data sets.

MGSE9-12.S.ID.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

KEY IDEAS

1. Two ***measures of central tendency*** that help describe a data set are mean and median.
   - The ***mean*** is the sum of the data values divided by the total number of data values.
   - The ***median*** is the middle value when the data values are written in order from least to greatest. If a data set has an even number of data values, the median is the mean of the two middle values.
2. The ***first quartile***, or the ***lower quartile***, Q_1, is the median of the lower half of a data set.

   Example:

   Ray's scores on his mathematics tests were 70, 85, 78, 90, 84, 82, and 83. To find the first quartile of his scores, write them in order from least to greatest:

   70, 78, 82, 83, 84, 85, 90

   The scores in the lower half of the data set are 70, 78, and 82. The median of the lower half of the scores is 78.

   So, the first quartile is 78.
3. The ***third quartile***, or the ***upper quartile***, Q_3, is the median of the upper half of a data set.

Example:

Referring to the previous example, the upper half of Ray's scores is 84, 85, and 90. The median of the upper half of the scores is 85.

So, the third quartile is 85.

4. The ***interquartile range*** **(IQR)** of a data set is the difference between the third and first quartiles, or $Q_3 - Q_1$.

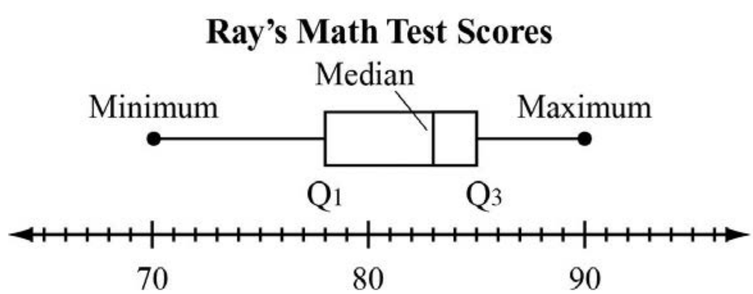
   Example:

   Referring again to the example of Ray's scores, to find the interquartile range, subtract the first quartile from the third quartile. The interquartile range of Ray's scores is 85 – 78 = 7.

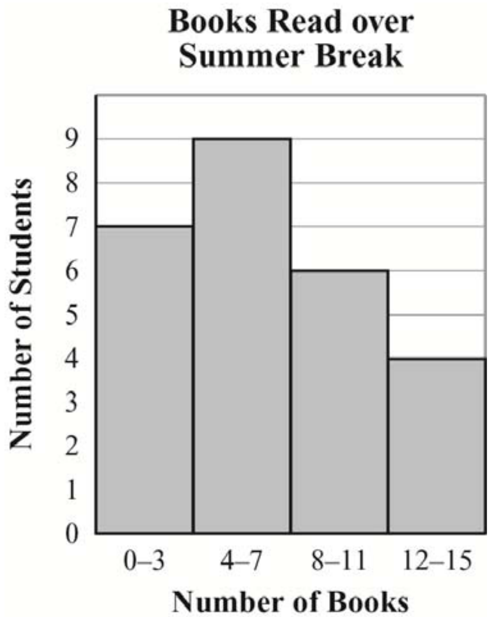
5. The most common displays for quantitative data are dot plots, histograms, box plots, and frequency distributions. A ***box plot*** is a diagram used to display a data set that uses quartiles to form the center box and the minimum and maximum to form the whiskers.

   Example:

   For the data in Key Idea 2, the box plot would look like the one shown below:



A ***histogram*** is a graphical display that subdivides the data into class intervals, called ***bins***, and uses a rectangle to show the frequency of observations in those intervals—for example, you might use intervals of 0–3, 4–7, 8–11, and 12–15 for the number of books students read over summer break.



6. Sometimes, distributions are characterized by extreme values that differ greatly from the other observations. These extreme values are called ***outliers***. A data value is an outlier if it is less than $Q_1 - 1.5 \cdot \text{IQR}$ or above $Q_3 + 1.5 \cdot \text{IQR}$.

   Example:

   This example shows the effect that an outlier can have on a measure of central tendency.

   The mean is one of several measures of central tendency that can be used to describe a data set. The main limitation of the mean is that, because every data value directly affects the result, it can be affected greatly by outliers. Consider these two sets of quiz scores:

   Student P: {8, 9, 9, 9, 10}
   Student Q: {3, 9, 9, 9, 10}

   Both students consistently performed well on quizzes, and both have the same median and mode score, 9. Student Q, however, has a mean quiz score of 8, while student P has a mean quiz score of 9. Although many instructors accept the use of a mean as being fair and representative of a student's overall performance in the context of test or quiz scores, it can be misleading because it fails to describe the variation in a student's scores, and the effect of a single score on the mean can be disproportionately large, especially when the number of scores is small.

7. ***Mean absolute deviation*** is the distance each data value is from the mean of the data set. This helps to get a sense of how spread out a data set is.

Example:

This example shows two sets of data that have the same mean but different mean absolute deviations. Consider the quiz scores of two students:

Student R: {3, 6, 8, 8, 9, 10, 12}
Student S: {1, 1, 3, 7, 14, 15, 15}

The mean score of student R is 8, and the mean score of student S is also 8. Determining the mean does not provide us with which student was more consistent. Which student is more consistent is what the mean absolute deviation will provide. We can use this formula

$$\frac{\sum_{i=1}^{n} \left| x_i - \overline{x} \right|}{n}$$

to find the mean absolute deviation. To apply the formula, we need to find the sum of the difference of the terms and the mean $\sum_{i=1}^{n} \left| x_i - \overline{x} \right|$. So,

Student R: $|3 - 8| + |6 - 8| + |8 - 8| + |8 - 8| + |9 - 8| + |10 - 8| + |12 - 8| = 14$
Student S: $|1 - 8| + |1 - 8| + |3 - 8| + |7 - 8| + |14 - 8| + |15 - 8| + |15 - 8| = 40$

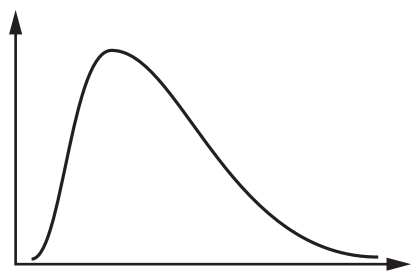
The final step is to divide the sums by the number of data, n.

Student R: $\frac{14}{8} = 1.75$ Student S: $\frac{40}{8} = 5$

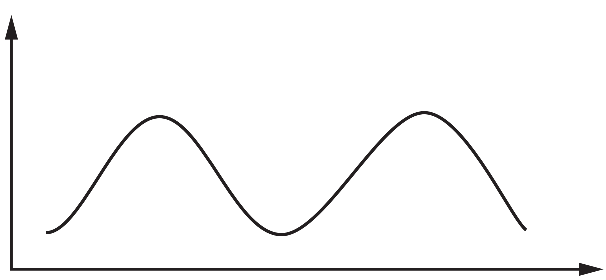
Since the mean absolute deviation of student R is smaller than the mean absolute deviation of student S, this means the quiz scores of student R were more consistent.

8. ***Skewness*** refers to the type and degree of a distribution's asymmetry. A distribution is skewed to the left if it has a longer tail on the left side and has a negative value for its skewness. If a distribution has a longer tail on the right, it has positive skewness. Generally, distributions have only one peak, but there are distributions called ***bimodal*** or ***multimodal*** where there are two or more peaks, respectively. A distribution can have symmetry but not be a normal distribution. It could be too flat (uniform) or too spindly. A box plot can present a fair representation of a data set's distribution. For a normal distribution, the median should be very close to the middle of the box and the two whiskers should be about the same length.

Skewed to the left



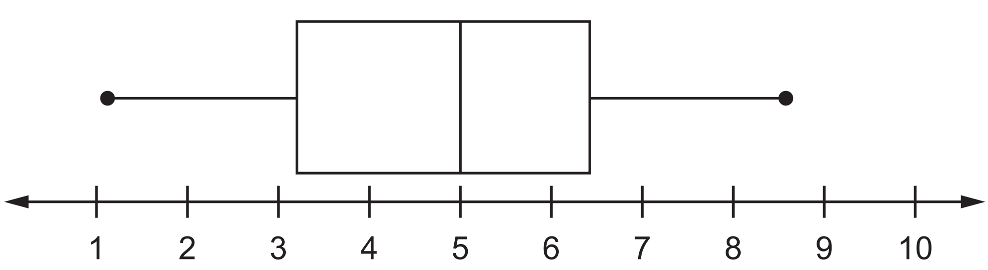
Skewed to the right



Bimodal representation

Important Tip

✍ The extent to which a data set is distributed normally can be determined by observing its skewness. Most of the data should lie in the middle near the median. The mean and the median should be fairly close. The left and right tails of the distribution curve should taper off. There should be only one peak, and it should neither be too high nor too flat.



REVIEW EXAMPLES

1. Josh and Richard each earn tips at their part-time jobs. This table shows their earnings from tips for five days.

Total Tips by Day

Day	Josh's Tips	Richard's Tips
Monday	\$40	\$40
Tuesday	\$20	\$45
Wednesday	\$36	\$53
Thursday	\$28	\$41
Friday	\$31	\$28

a. Who had the greater median earnings from tips? What is the difference in the median of Josh's earnings from tips and the median of Richard's earnings from tips?

b. What is the difference in the interquartile range for Josh's earnings from tips and the interquartile range for Richard's earnings from tips?

Solution:

a. Write Josh's earnings from tips in order from the least to greatest. Then, identify the middle value.

\$20, \$28, **\$31**, \$36, \$40

Josh's median earnings from tips are \$31.

Write Richard's earnings from tips in order from the least to greatest. Then, identify the middle value.

\$28, \$40, **\$41**, \$45, \$53

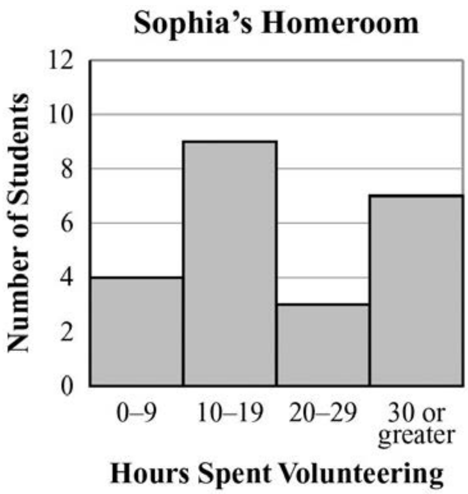
Richard had the greater median earnings from tips. The difference in the median of the earnings from tips is \$41 – \$31 = \$10.

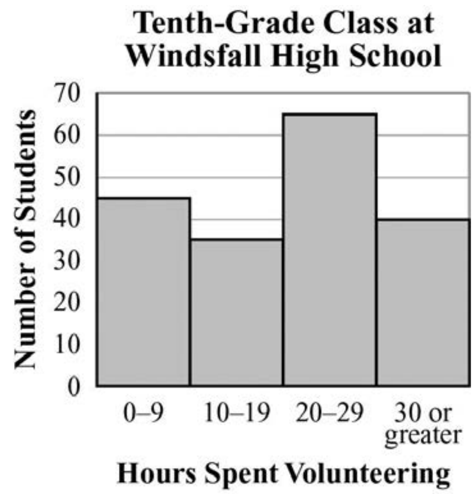
b. For Josh's earnings from tips, the lower quartile is \$24 and the upper quartile is \$38. The interquartile range is \$38 – \$24, or \$14.

For Richard's earnings from tips, the lower quartile is \$34 and the upper quartile is \$49. The interquartile range is \$49 – \$34, or \$15.

The difference in Josh's interquartile range and Richard's interquartile range is \$15 – \$14, or \$1.

2. Sophia is a student at Windsfall High School. These histograms give information about the number of hours spent volunteering by each of the students in Sophia's homeroom and by each of the students in the tenth-grade class at her school.





a. Compare the lower quartiles of the data in the histograms.
b. Compare the upper quartiles of the data in the histograms.
c. Compare the medians of the data in the histograms.

Solution:

a. You can add the number of students given by the height of each bar to find that there are 23 students in Sophia's homeroom. The lower quartile is the median of the first half of the data. That would be found within the 10–19 hours interval.

   You can add the number of students given by the height of each bar to find that there are 185 students in the tenth-grade class. The lower quartile for this group is found within the 10–19 hours interval.

   The interval of the lower quartile of the number of hours spent volunteering by each student in Sophia's homeroom is the same as the interval of the lower quartile of the number of hours spent volunteering by each student in the tenth-grade class.

b. The upper quartile is the median of the second half of the data. For Sophia's homeroom, that would be found in the 30 or greater interval.

   For the tenth-grade class, the upper quartile is found within the 20–29 hours interval.

   The upper quartile of the number of hours spent volunteering by each student in Sophia's homeroom is greater than the upper quartile of the number of hours spent volunteering by each student in the tenth-grade class.

c. The median is the middle data value in a data set when the data values are written in order from least to greatest. The median for Sophia's homeroom is found within the 10–19 hours interval.

   The median for the tenth-grade class is found within the 20–29 hours interval.

   The median of the number of hours spent volunteering by each student in Sophia's homeroom is less than the median of the number of hours spent volunteering by each student in the tenth-grade class.

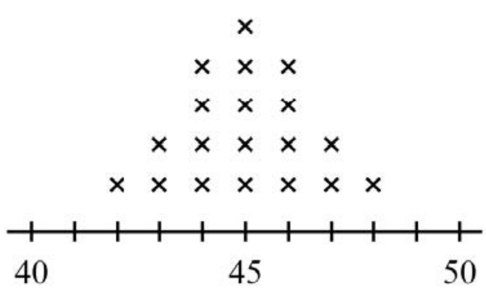
3. Mr. Storer, the physical education teacher, measured and rounded, to the nearest whole inch, the height of each student in his first-period class. He organized his data in this chart.

Height (inches)	Frequency
42	1
43	2
44	4
45	5
46	4
47	2
48	1

a. Make a dot plot for the data.
b. Make a histogram for the data.
c. Make a box plot for the data.

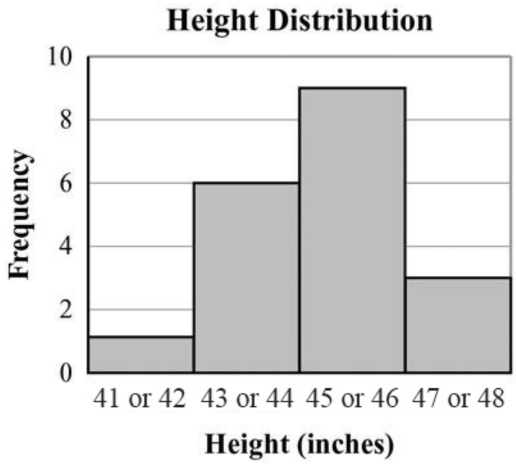
Solution:

a.

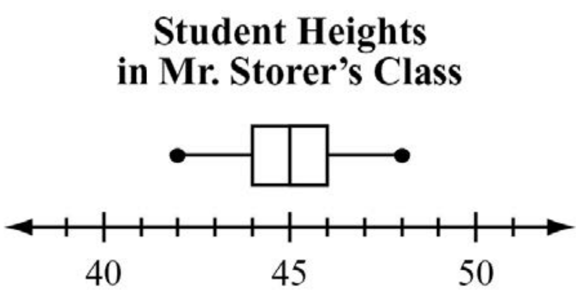


Student Heights in Mr. Storer's Class

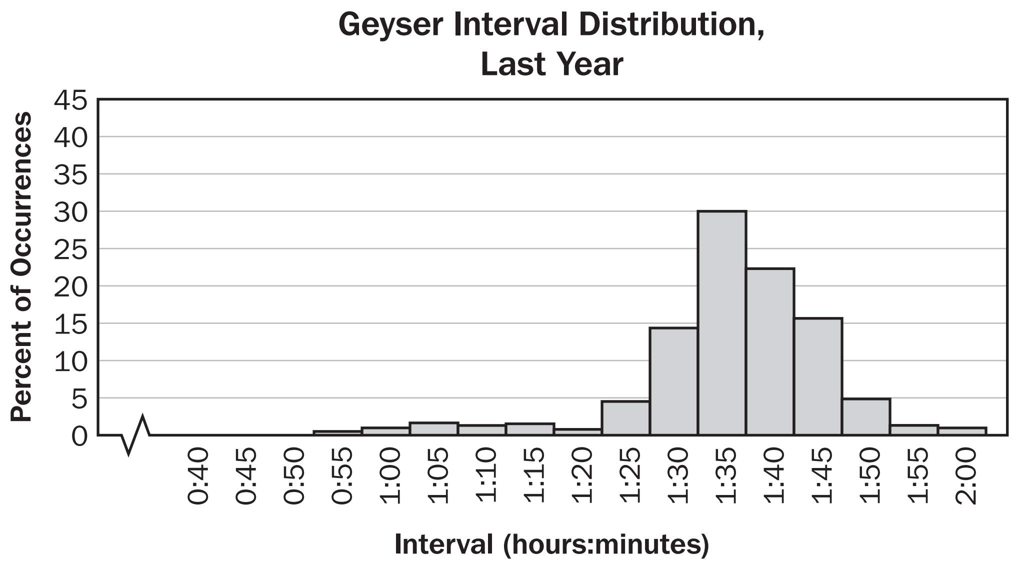
b.

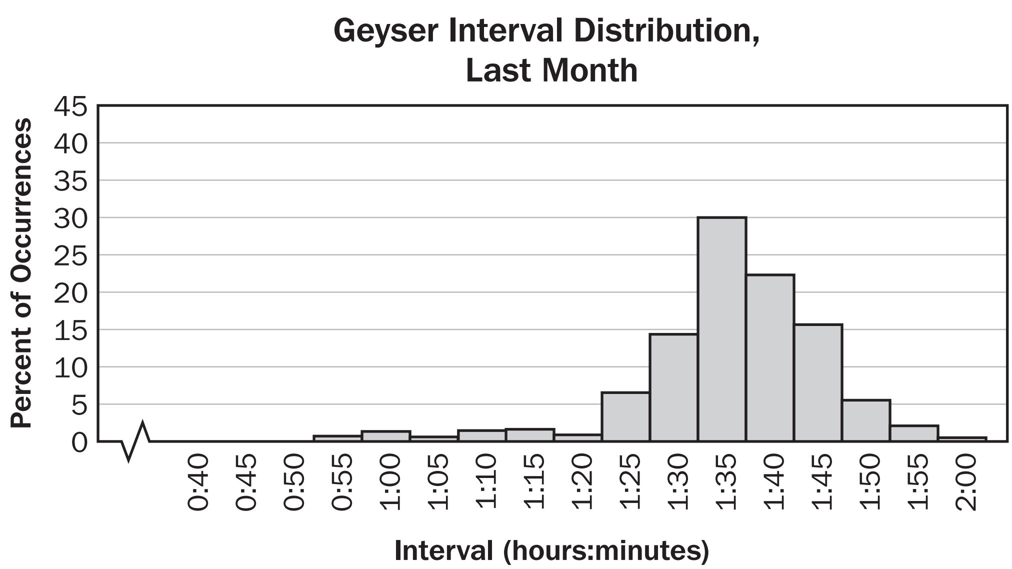

Height Distribution
Frequency
10
8
6
4
2
0
41 or 42
43 or 44
45 or 46
47 or 48
Height (inches)

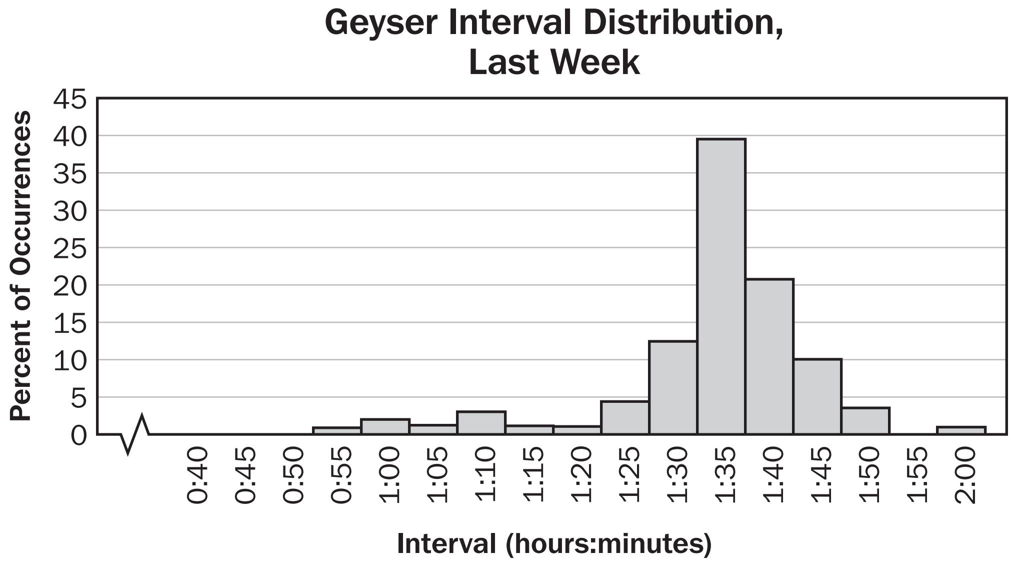
c.


Student Heights
in Mr. Storer's Class
40
45
50

4. A geyser in a national park erupts fairly regularly. In more recent times, it has become less predictable. It was observed last year that the time interval between eruptions was related to the duration of the most recent eruption. The distribution of its interval times for last year is shown in the following graphs.







a. Does the Last Year distribution seem skewed or uniform?
b. Compare Last Week's distribution to Last Month's distribution.
c. What does the Last Year distribution tell you about the interval of time between the geyser's eruptions?

Solution:

a. The Last Year distribution appears to be skewed to the left (negative). Most of the intervals approach 90 minutes.
b. Last Week's distribution seems more skewed to the left than Last Month's. It is also more asymmetric because of its high number of 1-hour-and-35-minute intervals between eruptions. Last Month's distribution appears to have the highest percentage of intervals longer than 1 hour 30 minutes between eruptions.
c. The Last Year distribution shows that the geyser rarely erupts an hour after its previous eruption. Most visitors will have to wait more than 90 minutes to see two eruptions.

SAMPLE ITEMS

1. This table shows the average low temperature, in °F, recorded in Macon, GA, and Charlotte, NC, over a six-day period.

Day	1	2	3	4	5	6
Temperature in Macon, GA (°F)	71	72	66	69	71	73
Temperature in Charlotte, NC (°F)	69	64	68	74	71	75

Which conclusion can be drawn from the data?

A. The interquartile range of the temperatures is the same for both cities.
B. The lower quartile for the temperatures in Macon is less than the lower quartile for the temperatures in Charlotte.
C. The mean and median temperatures in Macon were higher than the mean and median temperatures in Charlotte.
D. The upper quartile for the temperatures in Charlotte was less than the upper quartile for the temperatures in Macon.

Correct Answer: C

2. A school was having a coat drive for a local shelter. A teacher determined the median number of coats collected per class and the interquartile range of the number of coats collected per class for the freshmen and for the sophomores.

- **The freshmen collected a median number of coats per class of 10, and the interquartile range was 6.**
- **The sophomores collected a median number of coats per class of 10, and the interquartile range was 4.**

Which range of numbers includes the third quartile of coats collected for both freshmen and sophomore classes?

A. 4 to 14
B. 6 to 14
C. 10 to 16
D. 12 to 15

Correct Answer: C

3. A reading teacher recorded the number of pages read in an hour by each of her students. The numbers are shown below.

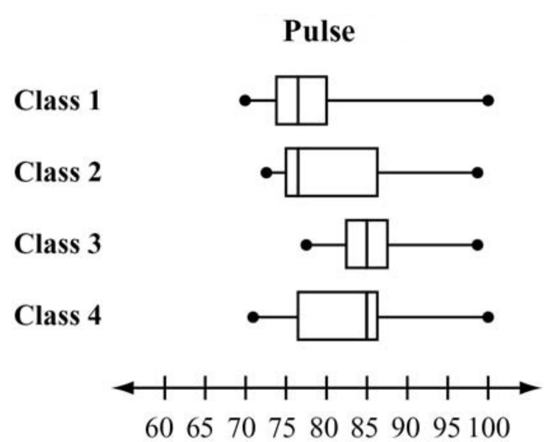
44, 49, 39, 43, 50, 44, 45, 49, 51

For this data, which summary statistic is NOT correct?

A. The minimum is 39.
B. The lower quartile is 44.
C. The median is 45.
D. The maximum is 51.

Correct Answer: B

4. A science teacher recorded the pulse of each of the students in her classes after the students had climbed a set of stairs. She displayed the results, by class, using the box plots shown.



Which class generally had the highest pulse after climbing the stairs?

A. Class 1
B. Class 2
C. Class 3
D. Class 4

Correct Answer: C

5. Peter went bowling, Monday to Friday, two weeks in a row. He only bowled one game each time he went. He kept track of his scores below.

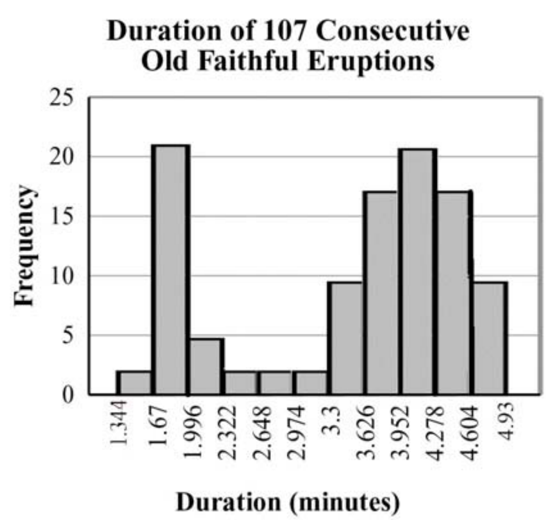
Week 1: 70, 70, 70, 73, 75
Week 2: 72, 64, 73, 73, 75

What is the BEST explanation for why Peter's Week 2 mean score was lower than his Week 1 mean score?

A. Peter received the same score three times in Week 1.
B. Peter had one very low score in Week 2.
C. Peter did not beat his high score from Week 1 in Week 2.
D. Peter had one very high score in Week 1.

Correct Answer: B

6. This histogram shows the frequency distribution of duration times for 107 consecutive eruptions of the Old Faithful geyser. The duration of an eruption is the length of time, in minutes, from the beginning of the spewing of water until it stops. What is the BEST description for the distribution?



A. bimodal
B. uniform
C. multiple outlier
D. skewed to the right

Correct Answer: A

Summarize, Represent, and Interpret Data on Two Categorical and Quantitative Variables

MGSE9-12.S.ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

MGSE9-12.S.ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

> **MGSE9-12.S.ID.6a** Decide which type of function is most appropriate by observing graphed data, charted data, or by analysis of context to generate a viable (rough) function of best fit. Use this function to solve problems in context. Emphasize linear, quadratic, and exponential models.
>
> **MGSE9-12.S.ID.6c** Using given or collected bivariate data, fit a linear function for a scatter plot that suggests a linear association.

KEY IDEAS

1. There are essentially two types of data: ***categorical*** and ***quantitative***. Examples of categorical data are color, type of pet, gender, ethnic group, religious affiliation, etc. Examples of quantitative data are age, years of schooling, height, weight, test score, etc. Researchers use both types of data but in different ways. Bar graphs and pie charts are frequently associated with categorical data. Box plots, dot plots, and histograms are used with quantitative data. The measures of central tendency (mean, median, and mode) apply to quantitative data. Frequencies can apply to both categorical and quantitative data.
2. ***Bivariate data*** consist of pairs of linked numerical observations, or frequencies of things in categories. Numerical bivariate data can be presented as ordered pairs and in any way that ordered pairs can be presented: as a set of ordered pairs, as a table of values, or as a graph on the coordinate plane.

   Categorical example: frequencies of gender and club memberships for 9th graders.

   A bivariate chart, or ***two-way frequency chart***, is often used with data from two categories. Each category is considered a variable, and the categories serve as labels in the chart. Two-way frequency charts are made of cells. The number in each cell is the frequency of things that fit both the row and column categories for the cell. From the two-way frequency chart on the next page, we see that there are 12 males in the band and 3 females in the chess club.

Participation in School Activities			
School Club	**Gender**		
	Male	**Female**	**Totals**
Band	12	21	33
Chorus	15	17	32
Chess	16	3	19
Latin	7	9	16
Yearbook	28	7	35
Totals	78	57	**135**

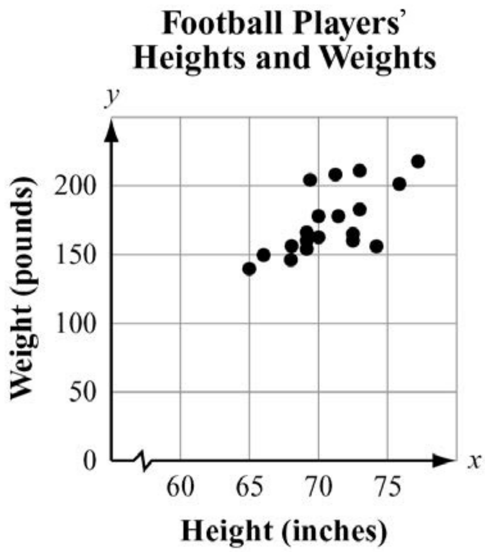
If no person or thing can be in more than one category per scale, the entries in each cell are called ***joint frequencies***. The frequencies in the cells and the totals tell us about the percentages of students engaged in different activities based on gender. For example, we can determine from the chart that if we picked at random from the students, we are least likely to find a female in the chess club because only 3 of 135 students are females in the chess club. The most popular club is yearbook, with 35 of 135 students in that club. The values in the table can be converted to percentages, which will give us an idea of the composition of each club by gender. We see that close to 14% of the students are in the chess club, and there are more than five times as many males as females.

Participation in School Activities			
School Club	**Gender**		
	Male	**Female**	**Totals**
Band	8.9%	15.5%	24.4%
Chorus	11.1%	12.6%	23.7%
Chess	11.9%	2.2%	14.1%
Latin	5.2%	6.7%	11.9%
Yearbook	20.7%	5.2%	25.9%
Totals	57.8%	42.2%	**100%**

There are also what we call ***marginal frequencies*** in the bottom and right margins (the shaded cells in the table above). These frequencies lack one of the categories. For our example, the frequencies at the bottom represent percentages of males and females in the school population. The marginal frequencies on the right represent percentages of club membership.

Lastly, associated with two-way frequency charts are ***conditional frequencies***. These are not usually in the body of the chart but can be readily calculated from the cell contents. One conditional frequency would be the percentage of chorus members that are female. The working condition is that the person is female. If 12.6% of the entire school population is females in the chorus and 42.2% of the student body is female, then 12.6% / 42.2%, or 29.9%, of the females in the school are in the chorus (also 17 of 57 females).

Quantitative example: Consider this chart of heights and weights of players on a football team.



A scatter plot is often used to present bivariate quantitative data. Each variable is represented on an axis, and the axes are labeled accordingly. Each point represents a player's height and weight. For example, one of the points represents a height of 66 inches and weight of 150 pounds. The scatter plot shows two players standing 70 inches tall because there are two dots on that height.

3. A ***scatter plot*** displays data as points on a grid using the associated numbers as coordinates. The way the points are arranged by themselves in a scatter plot may or may not suggest a relationship between the two variables. In the scatter plot about the football players shown previously, it appears there may be a relationship between height and weight because, as the players get taller, they seem to generally increase in weight; that is, the points are positioned higher as you move to the right. Bivariate data may have an underlying relationship that can be modeled by a mathematical function. Many of the examples in this review focus on linear models, but the models may take other forms, especially quadratic and exponential functions.

Example:

Melissa would like to determine whether there is a relationship between study time and mean test scores. She recorded the mean study time per test and the mean test score for students in three different classes.

This is the data for Class 1.

Class 1 Test Score Analysis	
Mean Study Time (hours)	**Mean Test Score**
0.5	63
1	67
1.5	72
2	76
2.5	80
3	85
3.5	89

Notice that, for these data, as the mean study time increases, the mean test score increases. It is important to consider the ***rate of increase*** when deciding which algebraic model to use. In this case, the mean test score increases by approximately 4 points for each 0.5-hour increase in mean study time. When the rate of increase is close to constant, as it is here, the best model is most likely a linear function.

This next table shows Melissa's data for Class 2.

Class 2 Test Score Analysis	
Mean Study Time (hours)	**Mean Test Score**
0.5	60
1	61
1.5	63
2	68
2.5	74
3	82
3.5	93

In these data as well, the mean test score increases as the mean study time increases. However, the rate of increase is not constant. The differences between each successive mean test score are 1, 2, 5, 6, 8, and 11.

This table shows Melissa's data for Class 3.

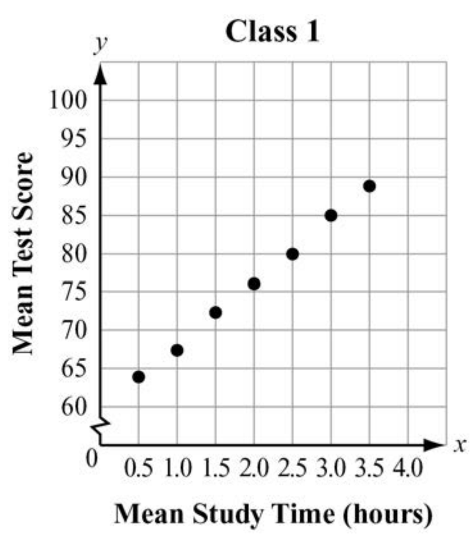
Class 3 Test Score Analysis	
Mean Study Time (hours)	Mean Test Score
0.5	71
1	94
1.5	87
2	98
2.5	69
3	78
3.5	91

In these data, as the mean study time increases, there is no consistent pattern in the mean test score. As a result, there does not appear to be any clear relationship between the mean study time and mean test score for this particular class.

Often, patterns in bivariate data are more easily seen when the data is plotted on a coordinate grid.

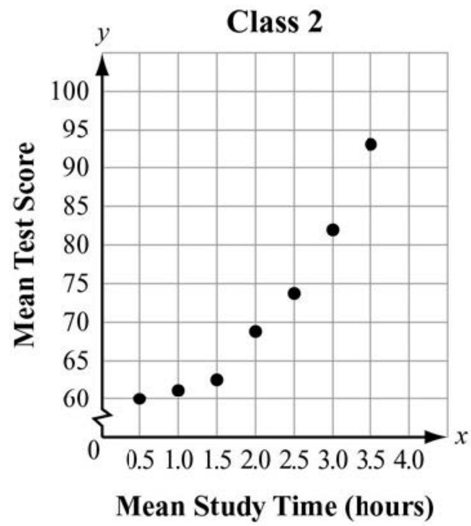
Example:

This graph shows Melissa's data for Class 1.



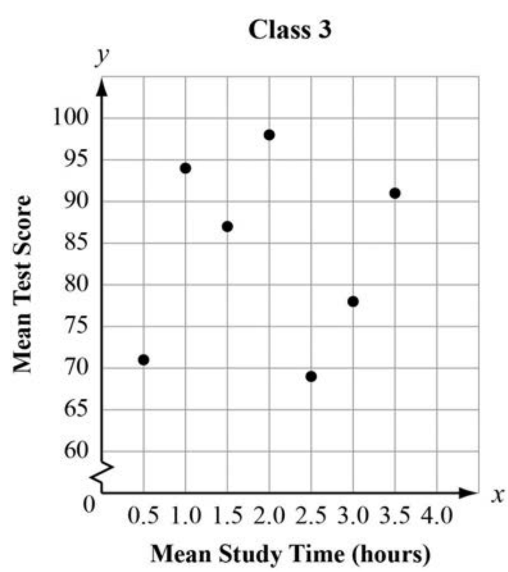
In this graph, the data points are all very close to being on the same line. This is further confirmation that a linear model is appropriate for this class.

This graph shows Melissa’s data for Class 2.



In this graph, the data points appear to lie on a curve, rather than on a line, with a rate of increase that increases as the value of *x* increases. It appears that a quadratic or exponential model may be more appropriate than a linear model for these data.

This graph shows Melissa’s data for Class 3.



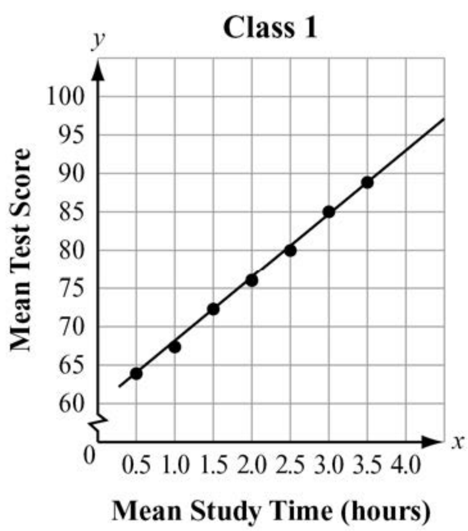
In this graph, the data points do not appear to lie on a line or on a curve. Linear, quadratic, and exponential models would not be appropriate to represent the data.

4. A ***line of best fit*** (trend or regression line) is a straight line that best represents the data on a scatter plot. This line may pass through some of the points, none of the points, or all of the points. In the previous examples, only the Class 1 scatter plot looks like a linear model would be a good fit for the points. In the other classes, a curved graph would seem to pass through more of the points. For Class 2, perhaps a quadratic model or an exponential model would produce a better-fitting curve. Since class 3 appears to have no correlation, creating a model may not produce the desired results.

   When a linear model is indicated there are several ways to find a function that approximates the *y*-value for any given *x*-value. A method called ***regression*** is the best way to find a line of best fit, but it requires extensive computations and is generally done on a computer or graphing calculator.

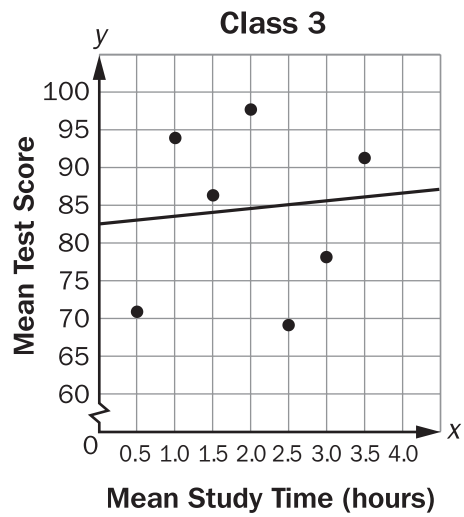
   Example:

   This graph shows Melissa's data for Class 1 with the line of best fit added. The equation of the line can be determined by using a calculator by entering the data into the STAT EDIT L1 and L2 in your calculator. L1 is for the *x*-values and L2 is for the *y*-values. Once all coordinates are entered, use the linear regression feature on the calculator. You will get values for a and b for the equation $y = ax + b$. For the data below, the equation is $y = 8.8x + 58.4$.



   Notice that five of the seven data points are on the line. This represents a very strong positive relationship for study time and test scores, since the line of best fit is positive and a very tight fit to the data points.

This next graph shows Melissa's data for Class 3 with the line of best fit added. The equation of the line is $y = 0.8x + 83.1$.

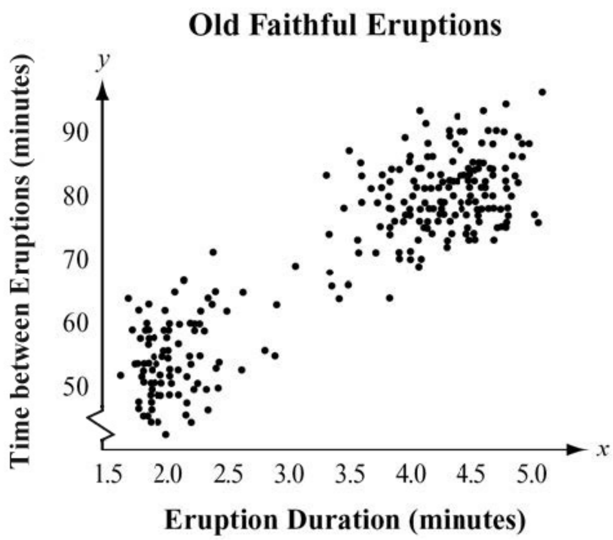


Although a line of best fit can be calculated for this set of data, notice that most of the data points are not very close to the line. In this case, although there is some correlation between study time and test scores, the amount of correlation is very small.

This is called the correlation coefficient, which is discussed in more detail in the next section about linear models.

REVIEW EXAMPLES

1. Barbara is considering visiting Yellowstone National Park. She has heard about Old Faithful, the geyser, and she wants to make sure she sees it erupt. At one time, it erupted just about every hour. That is not the case today. The time between eruptions varies. Barbara went on the Web and found a scatter plot of how long an eruption lasted compared to the wait time between eruptions. She learned that, in general, the longer the wait time, the longer the eruption lasts. The eruptions take place anywhere from 45 minutes to 125 minutes apart. They currently average 90 minutes apart.



   a. For an eruption that lasts 4 minutes, about how long would the wait time be for the next eruption?
   b. What is the shortest duration time for an eruption?
   c. Determine whether the scatter plot has a positive or a negative correlation and explain how you know.

Solution:

   a. After a 4-minute eruption, it would be between 75 and 80 minutes for the next eruption.
   b. The shortest eruptions appear to be a little more than 1.5 minutes (1 minute and 35 seconds).
   c. The scatter plot has a positive correlation because as the eruption duration increases, the time between eruptions increases.

2. The environment club is interested in the relationship between the number of canned beverages sold in the cafeteria and the number of cans that are recycled. The data they collected are listed in this chart.

Beverage Can Recycling

Number of Canned Beverages Sold	18	15	19	8	10	13	9	14
Number of Cans Recycled	8	6	10	6	3	7	5	4

Find an equation of a line of best fit for the data.

Solution:

Use a calculator by entering the data into the STAT EDIT L1 and L2 in your calculator. Enter the values of x in the L1 column and the values of y in the L2 column. Once all coordinates are entered, use the linear regression feature on the calculator. You will get values for a and b for the equation $y = ax + b$. For the data in the table, the equation is $y = 0.38x + 0.99$.

3. A fast-food restaurant wants to determine whether the season of the year affects the choice of soft-drink size purchased. It surveyed 278 customers, and the table below shows its results. The drink sizes were small, medium, large, and jumbo. The seasons of the year were spring, summer, and fall. In the body of the table, the cells list the number of customers who fit both row and column titles. On the bottom and in the right margin are the totals.

	Spring	Summer	Fall	TOTALS
Small	24	22	18	64
Medium	23	28	19	70
Large	18	27	29	74
Jumbo	16	21	33	70
TOTALS	81	98	99	278

a. In which season did the most customers prefer jumbo drinks?
b. What percentage of those surveyed purchased small drinks?
c. What percentage of those surveyed purchased medium drinks in the summer?
d. What do you think the fast-food restaurant learned from its survey?

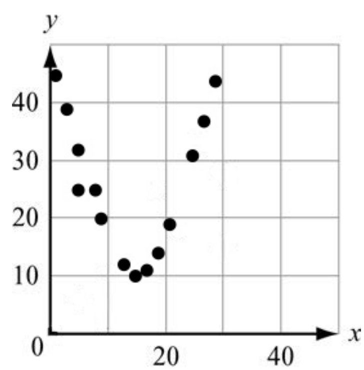
Solution:

a. The most customers preferred jumbo drinks in the fall.
b. Twenty-three percent (64/278 = 23%) of the 278 surveyed purchased small drinks.
c. Ten percent (28/278 = 10%) of those customers surveyed purchased medium drinks in the summer.
d. The fast-food restaurant probably learned that customers tend to purchase the larger drinks in the fall and the smaller drinks in the spring and summer.

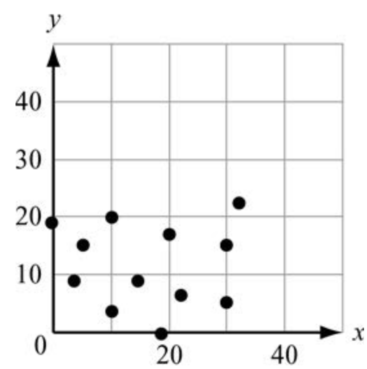
SAMPLE ITEMS

1. Which graph MOST clearly displays a set of data for which a quadratic function is the model of best fit?

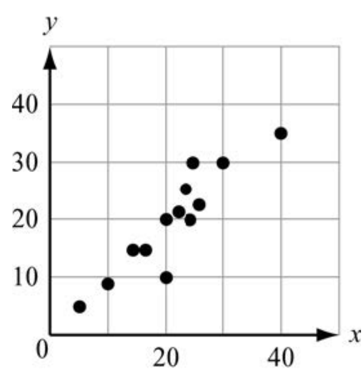
A.



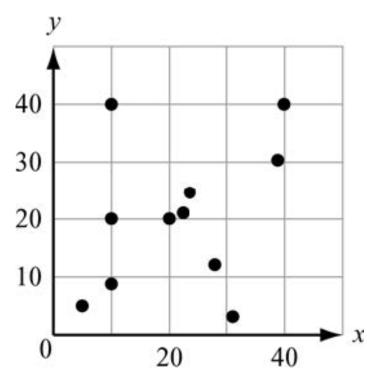
B.



C.

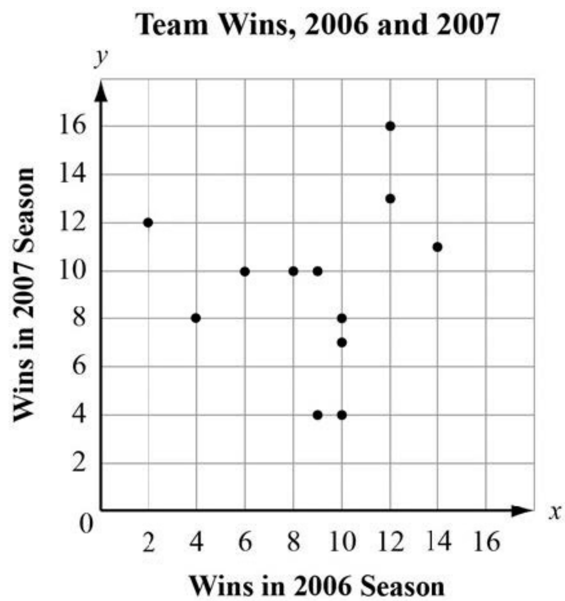


D.



Correct Answer: A

2. **This graph plots the number of wins in the 2006 season and in the 2007 season for a sample of professional football teams.**



Which equation BEST represents a line that matches the trend of the data?

A. $y = x + 2$

B. $y = x + 7$

C. $y = \frac{3}{5}x + 1$

D. $y = \frac{3}{5}x + 5$

Correct Answer: D

Interpret Linear Models

MGSE9-12.S.ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

MGSE9-12.S.ID.8 Compute (using technology) and interpret the correlation coefficient "r" of a linear fit. (For instance, by looking at a scatter plot, students should be able to tell if the correlation coefficient is positive or negative and give a reasonable estimate of the "r" value.) After calculating the line of best fit using technology, students should be able to describe how strong the goodness of fit of the regression is, using "r."

MGSE9-12.S.ID.9 Distinguish between correlation and causation.

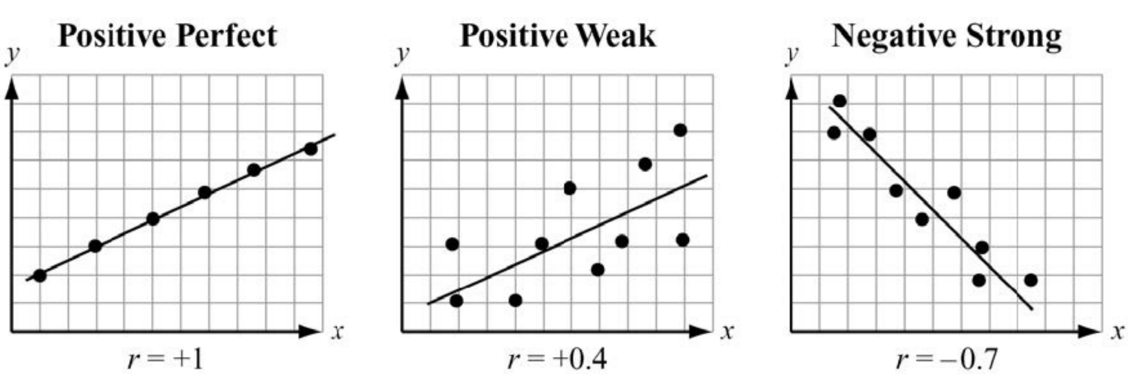
KEY IDEA

Once a model for the scatter plot is determined, we can begin to analyze the correlation of the linear fit. We can also interpret the slope, or rate of change, and the constant term and distinguish between correlation and causation of the data.

1. A ***correlation coefficient*** is a measure of the strength of the linear relationship between two variables. It also indicates whether the dependent variable, *y*, grows along with *x*, or *y* gets smaller as *x* increases. The correlation coefficient is a number between –1 and +1 including –1 and +1. The letter *r* is usually used for the correlation coefficient. When the correlation is positive, the line of best fit will have a positive slope and both variables are growing. However, if the correlation coefficient is negative, the line of best fit has a negative slope and the dependent variable is decreasing. The numerical value is an indicator of how closely the data points are modeled by a linear function.

   When using a calculator, use the same steps as you did to find the line of best fit. Notice there is a value, *r*, below the values for *a* and *b*. This is the correlation coefficient.

   Examples:



   The correlation between two variables is related to the slope and the goodness of the fit of a regression line. However, data in scatter plots can have the same regression lines and very different correlations. The correlation's sign will be the same as the slope of the regression line. The correlation's value depends on the dispersion of the data points and their proximity to the line of best fit.

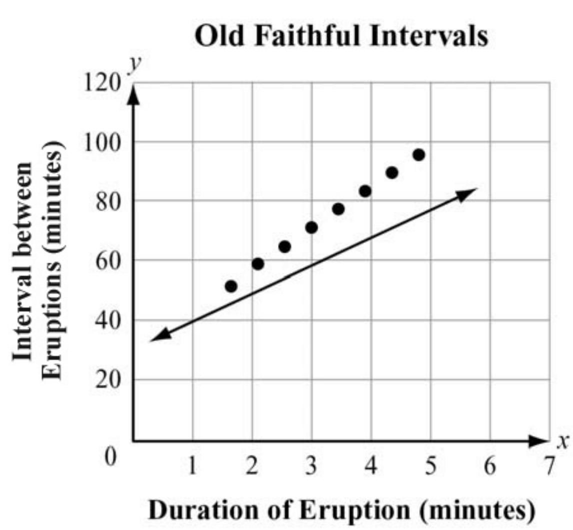
Example:

Earlier we saw that the interval between eruptions of Old Faithful is related to the duration of the most recent eruption. Years ago, the National Park Service had a simple linear equation they used to help visitors determine when the next eruption would take place. Visitors were told to multiply the duration of the last eruption by 10 and add 30 minutes ($I = 10 \cdot D + 30$). We can look at a 2011 set of data for Old Faithful, with eight data points, and see how well that line fits the 2011 data. The data points are from a histogram with intervals of 0.5 minute for *x*-values. The *y*-values are the average interval time for an eruption in that duration interval.

Old Faithful Eruptions

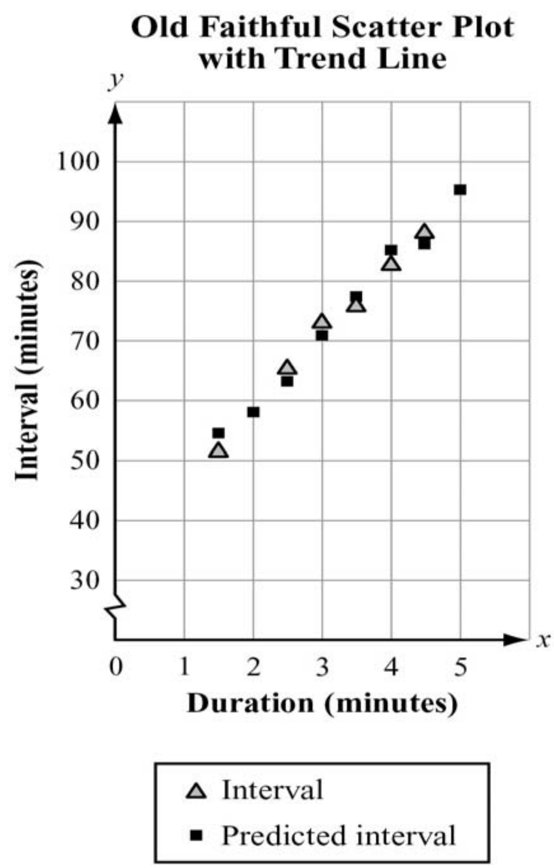
Duration (x)	Interval (y)	Prediction	Error Distances
1.50	51.00	45	6
2.00	58.00	50	8
2.50	65.00	55	10
3.00	71.00	60	11
3.50	76.00	65	11
4.00	82.00	70	12
4.50	89.00	75	14
5.00	95.00	80	15

The error distances display a clear pattern. The Park Service's regression line on the scatter plot shows the same reality. They keep increasing by small increments. The formula $I = 10 \cdot D + 30$ no longer works as a good predictor. In fact, it is a worse predictor for longer eruptions.



Instead of using the old formula, the National Park Service has a chart like the one in this example for visitors when they want to gauge how long it will be until the next eruption. We can take the chart the National Park Service uses and see what the new regression line would be. But first, does the scatter plot on the previous page look like we should use a linear model? And, do the *y*-values of the data points in the chart have roughly a constant difference?

The answer to both questions is “yes.” The data points do look as though a linear model would fit. The differences in intervals are all 5s, 6s, and 7s. In cases like this, you can use technology to find a linear regression equation by entering the data points in the STAT feature of your calculator.



The technology determines data points for the new trend line that appear to fit the observed data points much better than the old line. The interval-predicting equation has new parameters for the model, $a = 12.36$ (up 2.36 minutes) and $b = 33.2$ (up 3.2 minutes). The new regression line would be $y = 12.36x + 33.2$. While the new regression line appears to come much closer to the observed data points, there are still error distances, especially for lesser duration times. The scientists at Yellowstone Park believe that there probably should be two regression lines now: one for use with shorter eruptions and another for longer eruptions. As we saw from the frequency distribution earlier, Old Faithful currently tends to have longer eruptions that are farther apart.

The technology also provides a correlation coefficient. From the picture of the regression points on the previous page, it looks like the number should be positive and fairly close to 1. Using the linear regression feature on the calculator, we get $r = 0.9992$. Indeed, the length of the interval between Old Faithful's eruptions is very strongly related to its most recent eruption duration. The direction is positive, confirming the longer the eruption, the longer the interval between eruptions.

It is very important to point out that the length of Old Faithful's eruptions does not directly cause the interval to be longer or shorter between eruptions. The reason it takes longer for Old Faithful to erupt again after a long eruption is not technically known. However, with a correlation coefficient so close to 1, the two variables are closely related to one another. However, you should never confuse correlation with causation. For example, research shows a correlation between income and age, but aging is not the reason for an increased income. Not all people earn more money the longer they live. Variables can be related to each other without one causing the other.

Correlation is when two or more things or events tend to occur at about the same time and might be associated with each other but are not necessarily connected by a cause/effect relationship. ***Causation*** is when one event occurs as a direct result of another event. For example, a runny nose and a sore throat may correlate to each other but that does not mean a sore throat causes a runny nose or a runny nose causes a sore throat. Another example is it is raining outside and the ground being wet. There is a correlation between how wet the ground gets and how much it rains. In this case, the rain is what caused the ground to get more wet, so there is causation.

Example:

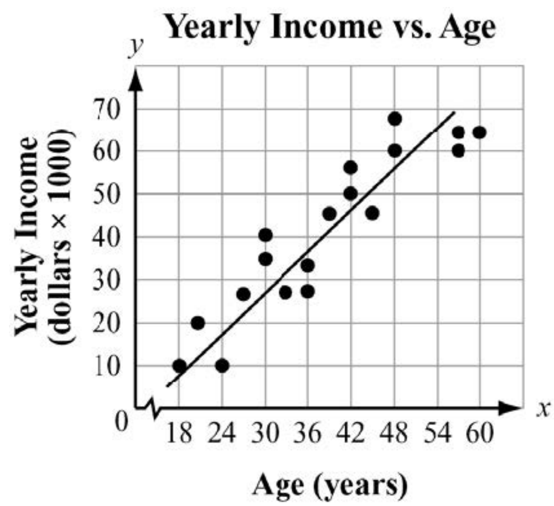
Consider the correlation between the age, in years, of a person and the income, in dollars, each person earns in the table below.

Age (years)	25	30	35	40	45	50
Income (dollars)	30,000	37,000	43,000	39,000	53,000	54,000

It appears there is a correlation between age and income and that a person's income increases as the person gets older. This does not mean age causes a person's income.

REVIEW EXAMPLES

1. This scatter plot suggests a relationship between the variables age and income.

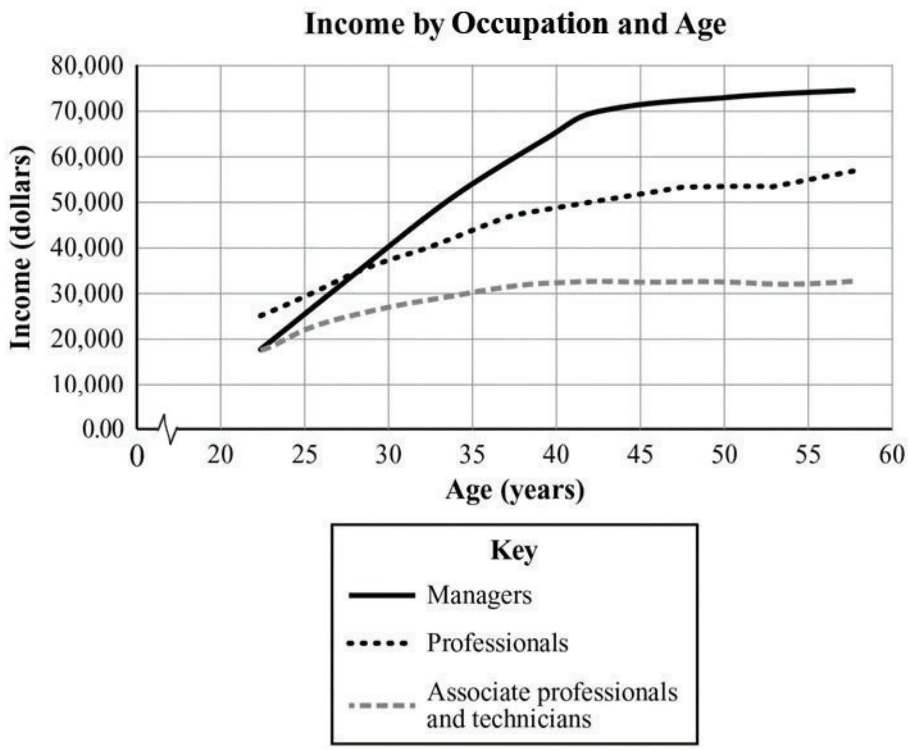


a. What type of a relationship is suggested by the scatter plot (positive/negative, weak/strong)?
b. What is the domain of ages considered by the researchers?
c. What is the range of incomes?
d. Do you think age causes income level to increase? Why or why not?

Solution:

a. The scatter plot suggests a fairly strong positive relationship between age and yearly income.
b. The domain of ages considered is 18 to 60 years.
c. The range of incomes appears to be $10,000 to $70,000.
d. No; the variables are related, but age does not cause income to increase.

2. A group of researchers looked at income and age in Singapore. Their results are shown below. They used line graphs instead of scatter plots so they could consider the type of occupation of the wage earner.



a. Does there appear to be a relationship between age and income?
b. Do all three types of occupations appear to share the same benefit of aging when it comes to income?
c. Does a linear model appear to fit the data for any of the occupation types?
d. Does the relationship between age and income vary over a person's lifetime?

Solution:

a. Yes, as people get older their income tends to increase.
b. No. The incomes grow at different rates until age 40. For example, the managers' incomes grow faster than those of the other occupation types until age 40.
c. No. The rate of growth appears to vary for all three occupations, making a linear model unsuitable for modeling this relationship over a longer domain.
d. Yes, after about age 40, the income for each type of occupation grows slower than it did from age 22 to 40.

3. Consider the table below.

Age (years)	Yearly Income (dollars × 1,000)	Prediction (dollars × 1,000)	Error Distances
18	10	8	2
20	20	13	7
24	10	17	–7
27	27	23	4
30	35	29	6
32	28	32	–4
36	28	37	–9
39	45	40	5
42	50	49	1
45	45	50	–5

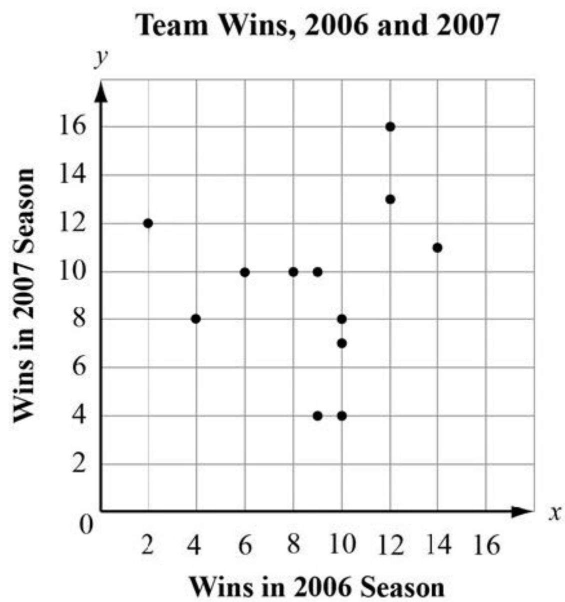
Do you think the regression line for this table is a good predictor of yearly income?

Solution:

The regression line for this table would be a good predictor of yearly income because the sum of the error differences is zero.

SAMPLE ITEMS

1. This graph plots the number of wins in the 2006 season and in the 2007 season for a sample of professional football teams.

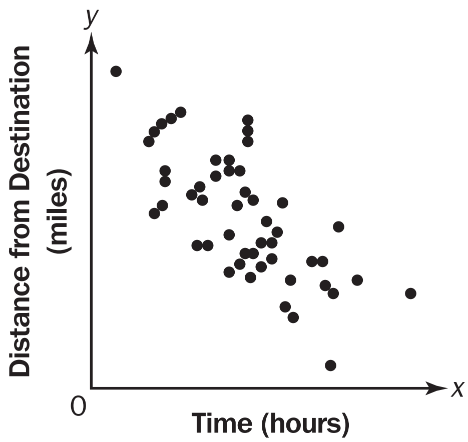


Based on the regression model, what is the predicted number of 2007 wins for a team that won 5 games in 2006?

A. 4
B. 7
C. 8
D. 12

Correct Answer: C

2. Which BEST describes the correlation of the two variables shown in the scatter plot?



A. weak positive
B. strong positive
C. weak negative
D. strong negative

Correct Answer: D

3. Which of these statements is an example of causation?

A. When the weather becomes warmer, more meat is purchased at the supermarket.
B. More people go to the mall when students go back to school.
C. The greater the number of new television shows, the fewer the number of moviegoers.
D. After operating costs are paid at a toy shop, as more toys are sold, more money is made.

Correct Answer: D

4. To rent a carpet cleaner at the hardware store, there is a set fee and an hourly rate. The rental cost, *c*, can be determined using this equation when the carpet cleaner is rented for *h* hours.

$$c = 25 + 3h$$

Which of these is the hourly rate?

A. 3
B. $3h$
C. 25
D. $25h$

Correct Answer: A

ALGEBRA I ADDITIONAL PRACTICE ITEMS

This section has two parts. The first part is a set of 18 sample items for Algebra I. The second part contains a table that shows for each item the standard assessed, the DOK level, the correct answer (key), and a rationale/explanation about the key and distractors. The sample items can be utilized as a mini-test to familiarize students with the item formats found on the assessment.

All example and sample items contained in this guide are the property of the Georgia Department of Education.

Georgia Milestones Assessment System

Algebra I Formula Sheet

You can find mathematics formula sheets on the Georgia Milestones webpage at http://www.gadoe.org/Curriculum-Instruction-and-Assessment/Assessment/Pages/Georgia-Milestones-Assessment-System.aspx.

Look under "EOC Resources."

Item 1

Sandra sells necklaces at a school craft fair. She uses the equation $P = 7.5n - (2.25n + 15)$ to determine her total profit at the fair. Based on this equation, how much does she charge for each necklace?

A. $2.25
B. $7.50
C. $15.00
D. $17.25

Item 2

The perimeter of a rectangle is $P = 2w + 2l$ where w is the width of the rectangle and l is the length of the rectangle. Rearrange this formula to find the width of the rectangle.

A. $w = P - 2l$

B. $w = \dfrac{P}{4 - l}$

C. $w = 2P - l$

D. $w = \dfrac{P}{2} - l$

Item 3

Read the following situation to determine whether the inequality correctly models the company's information.

The Mascot Company wants to spend no more than 1,250 dollars per month on the cost of school spirit items for sporting events. Production costs are 5 dollars per shirt and 8 dollars per banner. The company also wants monthly gross revenue from selling shirts and banners to be greater than 3,000 dollars. One shirt sells for 15 dollars and 1 banner sells for 20 dollars.

An employee at the company wants to determine the number of shirts and banners that Mascot Company should produce for a month. He lets *s* represent the number of shirts and *b* represent the number of banners. He writes the following system of inequalities.

$$\begin{cases} 5s + 8b \geq 1{,}250 \\ 15s + 20b > 3{,}000 \end{cases}$$

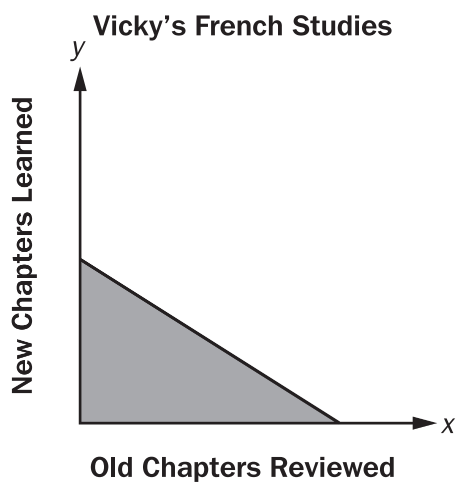
Part A: Does the first inequality correctly model the company's monthly costs? Explain. Write your answer on the lines provided.

Part B: Does the second inequality correctly model the company's monthly revenue? Explain. Write your answer on the lines provided.

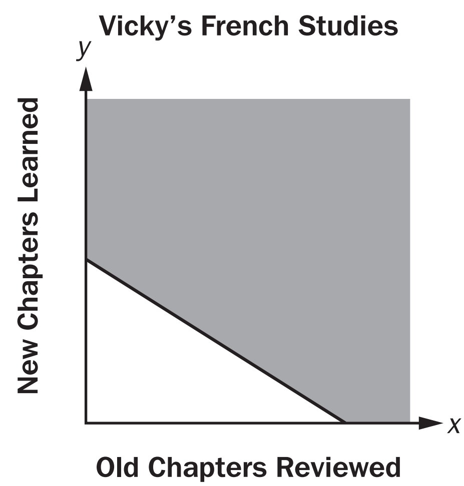
Item 4

Vicky is studying French. She spends 1 hour reviewing each old chapter. She also spends 1.5 hours learning each new chapter. She spends at least 10 hours per week studying French. Which graph could represent the possible number of old chapters Vicky reviews, *x*, and new chapters Vicky learns, *y*, each week?

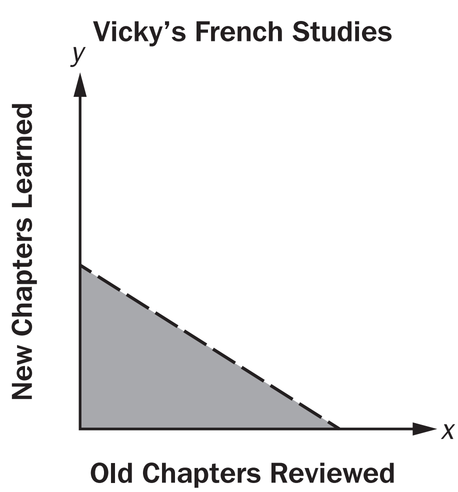
A.



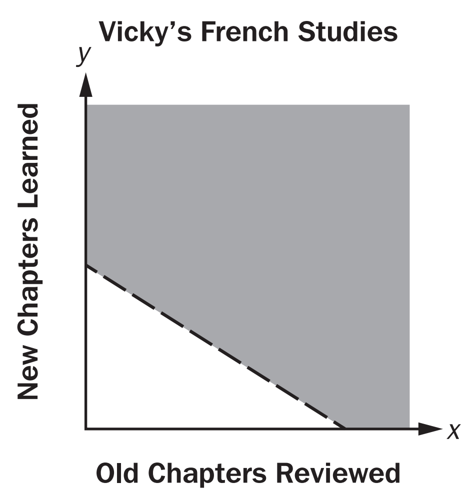
B.



C.



D.



Item 5

Kim uses these steps to solve a system of linear equations.

$$\begin{array}{lll} 2x - 3y = 2 & \rightarrow 4(2x - 3y = 2) & \rightarrow 8x - 12y = 8 \\ 8x - 12y = 5 & & \underline{-\,(8x - 12y = 5)} \\ & & \qquad\quad 0 = 3 \end{array}$$

Part A: Explain the steps Kim used to obtain her result. Write your answer on the lines provided.

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Part B: What can you conclude about the system of linear equations based on Kim's result? Write your answer on the lines provided.

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Item 6

It takes Matt 20 months to save $1,000.

Part A: Write an equation that models the average number of dollars, *x*, Matt saves each month. Write your answer on the lines provided.

Part B: How much money, in dollars, did Matt save each month? Write your answer on the lines provided.

Item 7

Which function can be used to model the data in this table?

x	$f(x)$
0	$^-1$
2	0
6	2

A. $f(x) = 3x$

B. $f(x) = \frac{x}{2} - 1$

C. $f(x) = x - 1$

D. $f(x) = 2x - 1$

Item 8

Amy owns a graphic design store. She purchases a new printer to use in her store. The printer depreciates by a constant rate of 14% per year. The function $V = 2{,}400(1 - 0.14)^t$ can be used to model the value of the printer in dollars after t years.

Part A: Explain what the parameter 2,400 represents in the equation of the function. Write your answer on the lines provided.

Part B: What is the factor by which the printer depreciates each year? Write your answer on the lines provided.

Part C: Amy also considered purchasing a printer that costs $4,000 and depreciates by 25% each year. Which printer will have more value in 5 years? Write your answer on the lines provided.

Part D: Amy wants to replace the printer after 6 years. She wants to sell her current printer and make a 150-dollar profit over the value of the printer after 6 years. At what price will she need to sell the printer to make a 150-dollar profit on the sale? Round your answer to the nearest dollar. Write your answer on the lines provided.

Item 9

The function $f(x) = x - 9$ is shifted 2 units up and 3 units to the left. Select the new function.

A. $g(x) = 2x - 6$
B. $g(x) = (x - 3) + 7$
C. $g(x) = 3x - 7$
D. $g(x) = (x + 3) - 7$

Item 10

A scientist studied the relationship between the number of trees, x, per acre and the number of birds, y, per acre in a neighborhood. She modeled the relationship with a scatter plot and used the equation $y = 4 + 6x$ for the regression line. What is the meaning of the slope and y-intercept of this regression line?

A. The slope is 6. This means that the average number of birds per acre in an area with no trees is 6. The y-intercept is 4. This means that for every 1 additional tree, she can expect an average of 4 additional birds per acre.
B. The slope is 4. This means that for every 1 additional tree, she can expect an average of 4 additional birds per acre. The y-intercept is 6. The average number of birds per acre in an area with no trees is 6.
C. The slope is 6. This means that for every 1 additional tree, she can expect an average of 6 additional birds per acre. The y-intercept is 4. The average number of birds per acre in an area with no trees is 4.
D. The slope is 4. This means that the average number of birds per acre in an area with no trees is 4. The y-intercept is 6. This means that for every 1 additional tree, she can expect an average of 6 additional birds per acre.

Item 11

A random group of high school students was surveyed. Each student was asked whether it should be mandatory for all high school students to participate in a sport. The results are partially summarized in the two-way table.

	Agree	Disagree	No Opinion	Total
Freshman	53	12	7	
Sophomore	65	37	2	104
Junior	18	42	12	
Senior	56	67	4	
Total		158		375

In the freshman group, what percentage of students agree that it should be mandatory for all students to participate in a sport?

A. 14.1%
B. 22.6%
C. 53%
D. 73.6%

Item 12

Maria and Jeff collect data on the number of cars that pass through an intersection every Monday morning for 2 months. They record the findings as 78, 158, 63, 71, 56, 67, 75, and 64. They each use different methods to summarize the typical number of cars that pass through the intersection at the specified time and compare their findings. Jeff says that on average, 79 cars pass through the intersection each Monday morning. Maria disagrees and says that the mean cannot be used and uses the median instead to describe the number of cars that pass through the intersection on a given Monday morning. She says that 69 cars pass through the intersection.

Part A: Whose method BEST describes the center of the data?

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Part B: Justify your answer.

Item 13

Which value is an irrational number?

A. $4 + \sqrt{7}$

B. $\sqrt{2}\sqrt{8}$

C. $\dfrac{\sqrt{3}\sqrt{12}}{5}$

D. $\sqrt{3} - \sqrt{3}$

Item 14

The table defines a quadratic function.

x	y
–1	5
0	1
1	–1
3	1

What is the average rate of change between $x = -1$ and $x = 1$?

A. undefined

B. $-\dfrac{1}{3}$

C. –3

D. –4

Item 15

Part A: What are the zeros of the function $f(x) = x^2 - 6x + 8$? Explain how you determined your answer. Write your answer on the lines provided.

Part B: Arturo made an error when finding the minimum value of the function $g(x) = x^2 - 6x + 10$. His work is shown below.

$$g(x) = x^2 - 6x + 10$$

$$g(x) = (x^2 - 6x - 9) + 10 + 9$$

$$g(x) = (x - 3)^2 + 19$$

The vertex is (3, 19), so the minimum value is 19.

Describe the error that Arturo made. Then give the correct minimum value of the function. Write your answer on the lines provided.

Item 16

Shaun recycles bottles and cans. He earns 10 cents for each bottle he recycles and 5 cents for each can he recycles. After recycling a bag of bottles and cans, he gets a receipt that states he earned $12.75 and recycled a total of 210 bottles and cans. To determine the number of bottles and the number of cans he recycled, Shaun writes the system of equations below.

$$x + y = 210$$
$$10x + 5y = 1275$$

Part A: Explain how you know that *x* represents the number of bottles Shaun recycled. Write your answer on the lines provided.

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Part B: Shaun graphs lines to represent the equations in his system. What are the coordinates of the point where the 2 lines intersect? Write your answer on the lines provided.

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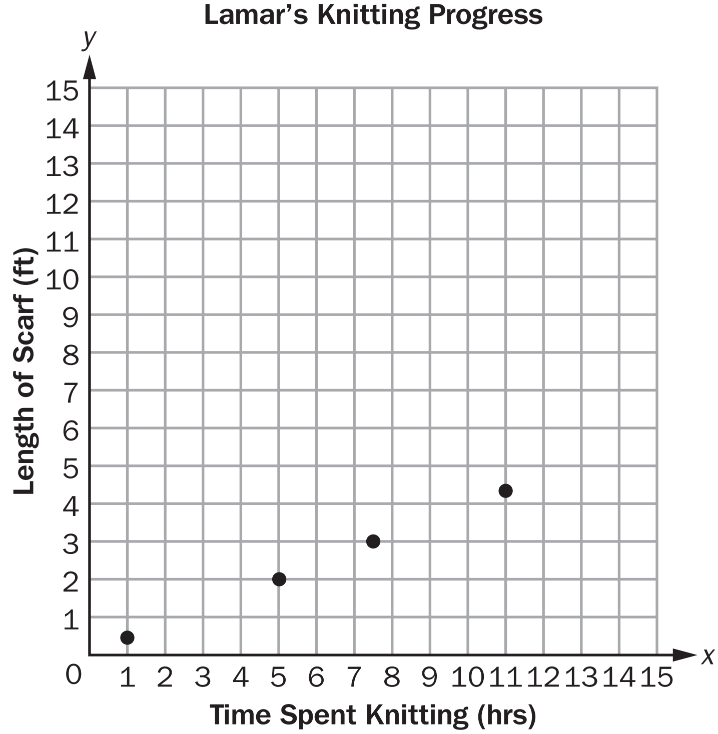
Item 17

The total area of two rectangles can be represented by the expression $(x)(3x + 1) + (2x)(x + 3)$. Which expression represents the total area of the two rectangles combined?

A. $7x^2$
B. $6x^3 + 6x^2$
C. $6x^2 + 7x$
D. $5x^2 + 7x$

Item 18

Lamar is knitting a scarf at a constant rate. He makes each row of the scarf 1 foot wide and finishes an entire row before starting the next row. At various times, he records how long he's been knitting and the length of the scarf. After knitting for a total of 11 hours, he records the length of his scarf. Then, he stops and makes this graph.



The finished scarf will be about 6 feet long and 1 foot wide. He estimates he is about 75% finished.

Part A: Lamar determines the rate at which he is knitting by calculating the slope of the graph. The slope of the graph is about 0.4. Explain why the unit rate for the graph could be 0.4 feet per hour. Explain why the unit rate for the graph could also be 0.4 square feet per hour. Write your answer on the lines provided.

Part B: Lamar decides to represent his unit rate in 0.4 feet per hour. Explain how he could convert his rate to inches per hour. Write your answer on the lines provided.

ADDITIONAL PRACTICE ITEMS ANSWER KEY

Item	Standard/ Element	DOK Level	Correct Answer	Explanation
1	MGSE9-12.A.SSE.1a	1	B	The correct answer is choice (B) $7.50 because profit equals 7.5 times the number of necklaces minus the cost per necklace and other fixed costs. Choice (A) is incorrect because it is the cost to produce each necklace. Choice (C) is incorrect because it is a fixed cost. Choice (D) is incorrect because it incorrectly adds the costs.
2	MGSE9-12.A.CED.4	2	D	The correct answer is choice (D) $w = \frac{P}{2 - l}$. To isolate w, divide both sides by 2 and then subtract l. Choices (A), (B), and (C) are incorrect because they incorrectly subtracted or divided to rearrange the formula to isolate w.
3	MGSE9-12.A.CED.3	3	N/A	See scoring rubric and exemplar responses on page 255.
4	MGSE9-12.A.REI.12	2	B	The correct answer is choice (B). The graph is correct because the line is solid and shaded above the line. Choices (A), (C), and (D) represent confusion over which inequality symbol to assign according to the graph.
5	MGSE9-12.A.REI.6	3	N/A	See scoring rubric and exemplar responses on page 256.
6	MGSE9-12.A.REI.3	2	N/A	See scoring rubric and exemplar responses on page 257.
7	MGSE9-12.F.IF.9	1	B	The correct answer is choice (B) $f(x) = \frac{x}{2} - 1$. By substituting the values of x into each function, only this one works for all values of x. Choices (A), (C), and (D) are incorrect because all values of x from the table will not give $f(x)$, even though some of the values might.
8	MGSE9-12.F.LE.5	3	N/A	See scoring rubric and exemplar responses beginning on page 258.

Item	Standard/ Element	DOK Level	Correct Answer	Explanation
9	MGSE9-12.F.BF.3	2	D	The correct answer is choice (D) $g(x) = (x + 3) - 7$. A shift up 2 units would subtract 2 units from the *y*-value, and a shift 3 units to the left would add 3 to the *x*-value. Choices (A) and (C) are incorrect because of errors of multiplying the *x*-value and subtracting from the *y*-value. Choice (B) is incorrect because the values are being subtracted instead of added to the *x*- and *y*-values.
10	MGSE9-12.S.ID.7	3	C	The correct answer is choice (C). Choice (A) is incorrect because the *y*-intercept is being used as a rate of change. Choice (B) has the incorrect interpretation of slope within the context. Choice (D) has the incorrect interpretation of slope and *y*-intercept within the context.
11	MGSE9-12.S.ID.5	2	D	The correct answer is choice (D) 73.6%. The conditional relative frequency of freshmen who agree is found by dividing 53 by the total number of freshmen surveyed, 72. Choice (A) is incorrect because 53 was divided by the total number of students surveyed, not just the freshmen. Choice (B) is incorrect because it calculated the ratio of freshmen who agreed to the number of freshmen who disagreed. Choice (C) is incorrect because it is the number of freshmen who agreed.
12	MGSE9-12.S.ID.2	3	N/A	See scoring rubric and exemplar responses on page 261.
13	MGSE9-12.N.RN.3	2	A	The correct answer is choice (A) because the sum is an irrational number. Choices (B), (C), and (D) are incorrect because they result in rational numbers.
14	MGSE9-12.F.IF.6	2	C	The correct answer is choice (C) because –3 is the slope of the line containing the indicated points. Choice (A) is incorrect because it reverses the numerator and denominator in the slope formula. Choice (B) and choice (D) are incorrect because there are arithmetic errors.
15	MGSE9-12.A.SSE.3a MGSE9-12.A.SSE.3b	3	N/A	See scoring rubric and exemplar responses beginning on page 262.
16	MGSE9-12.A.REI.6	3	N/A	See scoring rubric and exemplar responses on page 264.

Item	Standard/ Element	DOK Level	Correct Answer	Explanation
17	MGSE9-12.A.APR.1	2	D	The correct answer is choice (D) because the expression is equivalent to the expression representing the area of the rectangle. Choices (A), (B), and (C) are incorrect because the terms are combined incorrectly.
18	MGSE9-12.N.NQ.1	2	N/A	See scoring rubric and exemplar responses on page 265.

ADDITIONAL PRACTICE ITEMS SCORING RUBRICS AND EXEMPLAR RESPONSES

Item 3

Scoring Rubric

Points	Description
2	The response achieves the following: • Student gets both Part A and Part B correct.
1	The response achieves the following: • Student gets either Part A or Part B correct.
0	The response achieves the following: • Student gets neither Part A nor Part B correct.

Exemplar Response

Points Awarded	Sample Response
2	Part A: No. This inequality uses the wrong inequality symbol. It shows that the cost of the shirts plus the cost of the banners is at least 1,250 dollars, but the company wants to spend no more than 1,250 dollars on the costs. AND Part B: Yes. This inequality shows that the revenue from shirts plus the revenue from banners is greater than the company's revenue goal of 3,000 dollars.
1	Part A: No. This inequality uses the wrong inequality symbol. It shows that the cost of the shirts plus the cost of the banners is at least 1,250 dollars, but the company wants to spend no more than 1,250 dollars on the costs. OR Part B: Yes. This inequality shows that the revenue from shirts plus the revenue from banners is greater than the company's revenue goal of 3,000 dollars.
0	*Student does not produce a correct response or a correct process.*

Item 5

Scoring Rubric

Points	Description
2	The response achieves the following: • Student gets both Part A and Part B correct.
1	The response achieves the following: • Student gets either Part A or Part B correct.
0	The response achieves the following: • Student gets neither Part A nor Part B correct.

Exemplar Response

Points Awarded	Sample Response
2	Part A: Kim multiplied the first equation by 4. Then she multiplied the second equation by (–1) and added the two equations. The subtraction eliminated both variable terms on the left side, resulting in 0. The subtraction on the right side resulted in 3. The final equation is false because 0 is not equal to 3. AND Part B: Because the resulting equation is false, this system of linear equations has no solution.
1	Part A: Kim multiplied the first equation by 4. Then she multiplied the second equation by (–2) and added the two equations. The subtraction eliminated both variable terms on the left side, resulting in 0. The subtraction on the right side resulted in 3. The final equation is false because 0 is not equal to 3. OR Part B: Because the resulting equation is false, this system of linear equations has no solution.
0	*Student does not produce a correct response or a correct process.*

Item 6

Scoring Rubric

Points	Description
2	The response achieves the following: • Student gets both Part A and Part B correct.
1	The response achieves the following: • Student gets either Part A or Part B correct.
0	The response achieves the following: • Student gets neither Part A nor Part B correct.

Exemplar Response

Points Awarded	Sample Response
2	Part A: $1{,}000 = 20x$ AND Part B: \$50
1	Part A: $1{,}000 = 20x$ OR Part B: \$50
0	*Student does not produce a correct response or a correct process.*

Item 8

Scoring Rubric

<table>
<tr><th>Points</th><th>Description</th></tr>
<tr><td>4</td><td>The response achieves the following:
Student demonstrates complete and thorough understanding of interpreting the parameters in an exponential function in terms of a context. Award 4 points for a student response that contains all of the following elements:
Part A: Student states that the parameter 2,400 represents the initial value (or purchase price) of the printer in dollars.
Part B: 0.86
Part C: The printer that costs $2,400 will be worth $180 more than the printer that originally cost $4,000.
Part D: The discount does not affect the depreciation factor of the printer since the cost of the printer is still $2,400 and the depreciation percentage is still 14%.</td></tr>
<tr><td>3</td><td>The response achieves the following:
Student demonstrates nearly complete understanding of interpreting the parameters in an exponential function in terms of a context. Award 3 points for a student response that contains any 3 of the following elements:
Part A: Student states that the parameter 2,400 represents the initial value (or purchase price) of the printer in dollars.
Part B: 0.86
Part C: The printer that costs $2,400 will be worth $180 more than the printer that originally cost $4,000.
Part D: The discount does not affect the depreciation factor of the printer since the cost of the printer is still $2,400 and the depreciation percentage is still 14%.</td></tr>
<tr><td>2</td><td>The response achieves the following:
Student demonstrates partial understanding of interpreting the parameters in an exponential function in terms of a context. Award 2 points for a student response that contains any 2 of the following elements:
Part A: Student states that the parameter 2,400 represents the initial value (or purchase price) of the printer in dollars.
Part B: 0.86
Part C: The printer that costs $2,400 will be worth $180 more than the printer that originally cost $4,000.
Part D: The discount does not affect the depreciation factor of the printer since the cost of the printer is still $2,400 and the depreciation percentage is still 14%.</td></tr>
</table>

<table>
<tr><td>1</td><td>The response achieves the following:
Student demonstrates minimal understanding of interpreting the parameters in an exponential function in terms of a context. Award 1 point for a student response that contains any 1 of the following elements:
Part A: Student states that the parameter 2,400 represents the initial value (or purchase price) of the printer in dollars.
Part B: 0.86
Part C: The printer that costs $2,400 will be worth $180 more than the printer that originally cost $4,000.
Part D: The discount does not affect the depreciation factor or the printer since the cost of the printer is still $2,400 and the depreciation percentage is still 14%.</td></tr>
<tr><td>0</td><td>The response achieves the following:
The student demonstrates little to no understanding of interpreting the parameters in an exponential function in terms of a context.</td></tr>
</table>

Item 8

Exemplar Response

Points Awarded	Sample Response
4	Part A: The parameter 2,400 represents the initial value of the printer in dollars. Part B: 0.86 Part C: The printer that costs \$2,400 will be worth \$180 more than the printer that originally costs \$4,000. Part D: The discount does not affect the depreciation factor of the printer since the cost of the printer is still \$2,400 and the depreciation percentage is still 14%.
3	Part A: The parameter 2,400 represents the initial value of the printer in dollars. Part B: 0.86 Part C: The printer that costs \$2,400 will be worth \$180 more than the printer that originally costs \$4,000. Part D: This discount makes the printer worth \$50 less.
2	Part A: The parameter 2,400 represents the initial value of the printer in dollars. Part B: 0.86 Part C: The value of the printer that costs \$4,000 will be the same as the value of the printer that costs \$2,400 in the same number of years. Part D: This discount makes the printer worth \$50 less.
1	Part A: The parameter 2,400 represents the initial value of the printer in dollars. Part B: 14 Part C: The value of the printer that costs \$4,000 will be the same as the value of the printer that costs \$2,400 in the same number of years. Part D: This discount makes the printer worth \$50 less.
0	Part A: The parameter 2,400 represents the decrease in dollars in the value of the printer each year. Part B: 14 Part C: The value of the printer that costs \$4,000 will be the same as the value of the printer that costs \$2,400 in the same number of years. Part D: This discount makes the printer worth \$50 less.

Item 12

Scoring Rubric

Points	Description
2	The response achieves the following: • Student gets both Part A and Part B correct.
1	The response achieves the following: • Student response to Part A is incorrect; the response in Part B clearly acknowledges that 158 is an outlier and skews the mean, but it does not state the conclusion that Maria's method is correct for dealing with this situation.
0	The response achieves the following: • Student gets neither Part A nor Part B correct.

Exemplar Response

Points Awarded	Sample Response
2	Part A: Maria is correct. AND Part B: 158 is an outlier and skews the mean. That is why the center of the data is higher when Jeff found the average. Because Maria noticed an outlier, she realized that using the median would best describe the center of the data since median is not affected by the outlier.
1	Part A: Maria is correct. OR Part B: 158 is an outlier and skews the data, so the median that Maria found is lower than the mean that Jeff found.
0	*Student does not produce a correct response or a correct process.*

Item 15

Scoring Rubric

Points	Description

4	The response achieves the following:
	• Response demonstrates complete understanding of rewriting a quadratic function to find different properties. Award 4 points for a student response that contains all four of the following elements:
	  • states that the zeros of the function in Part A are 2 and 4
	  • explains how the zeros were determined
	  • identifies that the error in Part B involves the wrong values being used to complete the square and to keep the equation balanced
	  • states that the minimum of the function in Part B is 1
	Scoring Note: There are other valid ways of finding the zeros of the function in Part A. Accept any valid method.

3	The response achieves the following:
	• Response demonstrates nearly complete understanding of rewriting a quadratic function to find different properties. Award 3 points for a student response that contains three of the following elements:
	  • states that the zeros of the function in Part A are 2 and 4
	  • explains how the zeros were determined
	  • identifies that the error in Part B involves the wrong values being used to complete the square and to keep the equation balanced
	  • states that the minimum of the function in Part B is 1
	Scoring Note: There are other valid ways of finding the zeros of the function in Part A. Accept any valid method.

2	The response achieves the following:
	• Response demonstrates partial understanding of rewriting a quadratic function to find different properties. Award 2 points for a student response that contains two of the following elements:
	  • states that the zeros of the function in Part A are 2 and 4
	  • explains how the zeros were determined
	  • identifies that the error in Part B involves the wrong values being used to complete the square and to keep the equation balanced
	  • states that the minimum of the function in Part B is 1
	Scoring Note: There are other valid ways of finding the zeros of the function in Part A. Accept any valid method.

<table>
<tr><th>Points</th><th>Description</th></tr>
<tr><td>1</td><td>The response achieves the following:
Response demonstrates a minimal understanding of rewriting a quadratic function to find different properties. Award 1 point for a student response that contains only one of the following elements:
states that the zeros of the function in Part A are 2 and 4
explains how the zeros were determined
identifies that the error in Part B involves the wrong values being used to complete the square and to keep the equation balanced
states that the minimum of the function in Part B is 1
Scoring Note: There are other valid ways of finding the zeros of the function in Part A. Accept any valid method.</td></tr>
<tr><td>0</td><td>The response achieves the following:
Response demonstrates limited to no understanding of rewriting a quadratic function to find different properties.</td></tr>
</table>

Exemplar Response

Points Awarded	Sample Response
4	Part A: The zeros are 2 and 4. To find the zeros, I set the value of the function equal to 0. Then I factored the quadratic expression on the right side of the equation. Next, I used the Zero Product Property to set each factor equal to 0. Then I solved each of the resulting equations for *x*. These values of *x* are the zeros of the function. Part B: To complete the square, Arturo should have added 9 inside the parentheses instead of subtracting 9. And to keep the equation balanced, he should have subtracted 9 outside the parentheses instead of adding 9. The correct minimum value of the function is 1.
3	Part A: The zeros are 2 and 4. To find the zeros, I set the value of the function equal to 0. Then I factored the quadratic expression on the right side of the equation. Next, I used the Zero Product Property to set each factor equal to 0. Then I solved each of the resulting equations for *x*. These values of *x* are the zeros of the function. Part B: The correct minimum value of the function is 1.
2	Part A: The zeros are 2 and 4. Part B: The correct minimum value of the function is 1.
1	Part A: The zeros are 2 and 4. Part B: The correct minimum value of the function is 3.
0	*Student does not produce a correct response or a correct process.*

Item 16

Scoring Rubric

Points	Description

2	The response achieves the following:
	• Student demonstrates full understanding of solving a system of equations. Award 2 points for a student response that contains both of the following elements:
	  • states how to identify that the *x* variable represents the number of bottles recycled
	  • provides the correct coordinate of where the lines intersect

1	The response achieves the following:
	• Student shows partial understanding of solving a system of equations. Award 1 point for a student response that contains only one of the following elements:
	  • states how to identify that the *x* variable represents the number of bottles recycled
	  • provides the correct coordinate of where the lines intersect

0	The response achieves the following: • Student demonstrates little to no understanding of solving a system of equations.

Exemplar Response

Points Awarded	Sample Response
2	The second equation has $10x$ and $5y$, so that represents the amount of money he gets from each type (bottle or can). Since x has a coefficient of 10 and Shaun earns 10 cents per bottle, x must represent the number of bottles. AND (45, 165)
1	(45, 165)
0	*Student does not produce a correct response or a correct process.*

Item 18

Scoring Rubric

Points	Description

2	The response achieves the following:
	• Student demonstrates full understanding of finding the zeros of a function. Award 2 points for a student response that contains both of the following elements:
	  • states that the function has 2 zeros
	  • provides a valid explanation of how the number of zeros was determined

1	The response achieves the following:
	• Student shows partial understanding of solving a system of equations. Award 1 point for a student response that contains only one of the following elements:
	  • states that the function has 2 zeros
	  • provides a valid explanation of how the number of zeros was determined

0	The response achieves the following: • Student demonstrates little to no understanding of solving a system of equations.

Exemplar Response

Points Awarded	Sample Response
2	2 zeros AND Because the function is a parabola, at most there are two zeros. I solved for $f(0)$ and found two values for x that make the function true. Therefore, there are 2 zeros for this function.
1	2 zeros
0	*Student does not produce a correct response or a correct explanation.*

Study/Resource Guide
for Students and Parents
Algebra I
End-of-Course